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Quasi-fiscal Deficit Financing and (Hyper) Inflation

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In the Argentine hyperinflations of 1989 and 1990, quasi-fiscal deficits were a major part of the problem. The Central Bank´s quasi-fiscal activities are financed directly by money printing but in some cases the monetary authority tries to sterilize the effect on the money supply by issuing debt or by increasing reserve requirements (it is not uncommon to pay interest on reserves when this happens). Thus, a new source of quasi-fiscal deficit arises, i.e. the interest payments on the Bank´s liabilities. When nominal interest rates are high and debt reaches unsustainable levels, the interest payments can take a life of their own leading to hyperinflation. The traditional explanation is that the Central Bank has to finance the quasi-fiscal deficit through the use of the inflation tax but as inflation increases money demand drops and there is a limit to how much revenue can be collected which is determined by a Laffer curve. Trying to finance a quasi-fiscal deficit beyond that limit (or any fiscal deficit for that matter) leads to hyperinflation. In this paper we demonstrate that very high inflation can arise even if money demand is perfectly inelastic with respect to inflation and the real value of interest payments is relatively low. The key insight is that if expected inflation is a function of the current state of the economy the Central Bank has an additional incentive to alter the future state which results in higher inflation today.

JEL: E31, E52, E62

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1) Introduction.

In the Argentine hyperinflations of 1989 and 1990, quasi-fiscal deficits were a major part of the problem as shown in Almansi and Rodríguez (1990) and Rodríguez (1993). By quasi-fiscal deficit we mean the losses incurred by the Central Bank (CB) that are originated by quasi-fiscal activities which monetary authorities sometimes undertake and go beyond the basic provision of high power money through open market purchases of foreign currency or government bonds. Quasi-fiscal activities take many shapes and colors and how they are conceptually defined and measured is a whole other issue in itself that goes beyond the scope of this paper. Some examples are bailouts of public and private institutions, insurance schemes on deposits and exchange rates, direct lending to private and public enterprises at subsidized rates, etc. For a more thorough discussion of the issue see Blejer and Cheasty (1991).

Quasi-fiscal activities are financed directly by money printing but in some cases the CB tries to sterilize the effect on the money supply by issuing debt or by increasing reserve requirements (it is not uncommon for the CB to pay interest on reserves when this happens). Thus, a new source of quasi-fiscal deficit arises, i.e. the interest payments on the CB´s liabilities. When nominal interest rates are high and the CB´s debt reaches unsustainable levels, the interest payments can take a life of their own leading to hyperinflation. The traditional explanation is that the CB has to finance the quasi-fiscal deficit through the use of the inflation tax but as inflation increases money demand drops and there is a limit to how much revenue can be collected which is determined by a Laffer curve. Trying to finance a quasi-fiscal deficit beyond that limit (or any fiscal deficit for that matter) leads to hyperinflation.

In this paper we demonstrate that very high inflation can arise even if money demand is perfectly inelastic with respect to inflation and the real value of interest payments is relatively low. We show how this can happen when the CB follows a discretionary monetary policy in which it tries to affect market expectations. To do so we present a model which captures the main features of the problem at hand and find the equilibrium under different institutional arrangements since, as expected, the optimal policy is not time consistent. We show that if the CB can commit to a particular course of action, i.e. once a policy is announced the monetary authority is then forced to follow it, then the optimal policy does not impose a big cost in terms of inflation (at least when compared to previous experiences from high inflation countries). If commitment is not an option then hyperinflation might be a very real scenario if nominal liabilities reach levels of two times the monetary base. We also show that in our model there is room for a type of policy, which we interpret as inflation targeting, in which the CB announces a particular time path for inflation. If market expectations are set accordingly then the CB has no incentive to deviate from the preannounced target which implies that this “second best” policy is time consistent.

The issue discussed in this paper is closely related to the literature on optimal taxation. Lucas and Stokey (1983) describe the optimal taxation problem in a world without capital and show how the optimal policy is time consistent in a real economy provided that the debt structure is rich enough. They conjecture that the same result does not apply in general to monetary economies. Álvarez, Kehoe and Neumeyer (2004) show how the time consistency of optimal policies can be extended to a wide range of monetary economies in which the Friedman is optimal. A crucial assumption for their result is that initial nominal liabilities are equal to zero. In their model unexpected inflation has no welfare costs so if nominal liabilities are not zero then the government could inflate them away if they are positive or use them to levy a lump sum tax if they are negative. They conjecture that if unexpected inflation has welfare effects then the necessary conditions for time consistency of the optimal policy are weaker. Their conjecture is proven right in Persson, Persson and Svensson (2006) which shows that the optimal policy is time consistent even if nominal liabilities are positive but less than the money supply. Nicolini (1998) showed that in a particular cash in advanced model the optimal monetary policy is time consistent depending on the intertemporal elasticity of substitution.

The main differences with the previous literature are: we only model the behavior of the CB and the inflation tax is the only source of financing, the objective of the CB is not to maximize social welfare but to minimize an *ad hoc* loss function similar to the one used in Barro and Gordon (1983), and the CB can only issue base money and nominal debt. We also focus on the case where the nominal liabilities are greater than the monetary base. Section 2 presents the model and finds the solution under commitment. One solution without commitment is presented in section 3 and we show how this particular equilibrium can easily lead to very high inflation rates. Section 4 we presents "equilibrium"[1] in the continuous time version without commitment which we think is related to an inflation targeting rule. In section 5 we present numerical solutions to the models in section 2, 3 and 4. Section 6 shows how the solution with commitment is equivalent to an equilibrium in which the CB´s debt is real, we argue that a swap of nominal debt for real debt might be an adequate policy for a CB that cannot commit. Section 7 summarizes the main results and concludes the paper.

[1] The term equilibrium is used in a very loose sense in this particular case as we will discuss later.

2) The Model.

At time t the CB has nominal liabilities $L(t)$ which consist of nominal debt, $D(t)$, and the monetary base $M(t)$[2],

$$L(t) = D(t) + M(t).$$

We assume that the CB has no assets whatsoever. Each period the CB has to pay interests on the debt and transfer some money to the treasury. These expenses are financed by printing new money or issuing new debt,

$$i(t)D(t) + G(t) = \dot{M}(t) + \dot{D}(t)$$

where $i(t)$ is the nominal interest rate and $G(t)$ is the transfer from the CB to the treasury. The transfer is a constant fraction g of the monetary base,

$$G(t) = gM(t).$$

The demand for base money grows at a constant rate $\alpha > 0$ and we assume that it is perfectly inelastic with respect to the nominal interest rate, i.e.

$$\frac{M(t)}{P(t)} = me^{\alpha t}$$

where $P(t)$ is the price level at t. The nominal interest rate satisfies the Fisher equation which implies that the nominal interest is equal to the real interest rate, r, plus the expected inflation rate, $\pi^e(t)$, i.e.

$$i(t) = r + \pi^e(t).$$

Define the real value of nominal liabilities (normalized to account for growth in the demand for base money) as

$$x(t) \equiv \frac{L(t)}{P(t)e^{\alpha t}}.$$

With the information presented we can derive the law of motion for x as

$$\dot{x}(t) = \big(r - \alpha + \pi^e(t) - \pi(t)\big)(x(t) - m) - (\pi(t) + \alpha - g)m$$

The first term is the real interest payment (α is subtracted because of our normalization) which applies to the interest bearing part of the nominal liabilities and the second term is the revenue from the inflation tax net of transfers to the treasury.

[2] A more general definition would be one in which the distinction is made among interest bearing liabilities and non interest bearing liabilities.

We depart from the literature on optimal taxation by postulating and *ad hoc* loss function for the CB which is similar to the one used in Barro 1983 but without the term that captures the effect of unemployment. The CB´s loss function is assumed to be of the form[3]

$$C = \int_0^{+\infty} e^{-\rho t} \frac{\pi(t)^2}{2} dt.$$

We impose the following restriction on the parameters

$$0 < \rho < r - \alpha;$$

if the real interest rate is greater than the sum of the discount and growth rates then real value of nominal liabilities would grow indefinitely or until some upper bound is reached.

a) Commitment in Continuous Time.

Assume that the CB has a mechanism that allows it to commit to a policy announced at $t = 0$. Given $x(0)$ the CB´s problem is to minimize its loss function subject to[4]

$$\dot{x}(t) = (r - \alpha)(x(t) - m) - (\pi(t) + \alpha - g)m$$

which is the law of motion for $x(t)$ once we account for the fact that agents have rational expectations, $\pi^e(t) = \pi(t)$ and it can commit to follow the optimal plan announced at $t = 0$ in all subsequent periods $t > 0$. The Hamiltonian for the optimal plan with commitment is then given by

$$H = \frac{\pi^2}{2} + \lambda[(r - \alpha)(x - m) - (\pi + \alpha - g)m].$$

The conditions for an optimum are

$$\pi = \lambda m,$$

$$\dot{\lambda} = \lambda\rho - \lambda(r - \alpha),$$

together with the law of motion for x and the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)x(t) = 0.$$

From the first two conditions we get

[3] It is straight forward to include an inflation target, $\bar{\pi} \neq 0$ so that the loss function is expressed in terms of $\pi(t) - \bar{\pi}$. We could also include a “one time” only penalty to allow for a discrete jump in the price level at $t = 0$.

[4] We also need to impose a borrowing limit to rule out Ponzi schemes. This limit can be arbitrarily large so that it never binds in equilibrium.

$$\dot{\pi} = (\rho + \alpha - r)\pi,$$

so the optimal path is described by system of two linear differential equations

$$\begin{bmatrix} \dot{\pi} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \rho + \alpha - r & 0 \\ -m & r - \alpha \end{bmatrix} \begin{bmatrix} \pi \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ (g - r)m \end{bmatrix}$$

with steady state values: $\pi^* = 0$ and $x^* = \left(\frac{r-g}{r-\alpha}\right) m$. The characteristic roots are: $\varepsilon_1 = \rho + \alpha - r < 0$ and $\varepsilon_2 = r - \alpha > 0$. The initial value of x is given and the initial value of π has to be chosen so that the system converges to the steady state (x^*, π^*). The complete solution is then given by

$$\pi(t) = (2(r - \alpha) - \rho)\frac{x(0) - x^*}{m} e^{-(r-\alpha-\rho)t}$$

and

$$x(t) = (x(0) - x^*)e^{-(r-\alpha-\rho)t} + x^*.$$

When $\alpha = g = 0$ the solution can be expressed as

$$\pi(t)m = (2r - \rho)(x(t) - m).$$

On the left hand side we have the revenue from the inflation tax. On the right hand side we have the interest bearing part of nominal liabilities, $x(t) - m$, multiplied by $2r - \rho$ which is equal to one interest payment of r plus one capital payment of $r - \rho$. The loss function for the CB given $x(0) = x$ is

$$C(x) = \frac{2(r - \alpha) - \rho}{2}\left(\frac{x - x^*}{m}\right)^2.$$

Higher nominal liabilities lead to more inflation but the relationship is linear which implies that in general one would need very high levels of debt in order to get very high inflation rates.

b) Commitment in Discrete Time.

In discrete time the CB starts period t with nominal liabilities L_{t-1} which are given by the sum of the monetary base and debt issued at $t - 1$

$$L_{t-1} = M_{t-1} + D_{t-1}.$$

The CB has to pay this liabilities and transfer resource G_t the treasury by printing money or issuing new zero coupon bonds with price $1/(1 + i_t)$,

$$L_{t-1} + G_t = M_t + \frac{D_t}{1 + i_t},$$

where i_t is the nominal interest rate between periods t and $t + 1$ and it is determined by the real interest rate r and the expected inflation between t and $t + 1$,

$$1 + i_t = (1 + r)(1 + \pi_{t+1}^e).$$

The money demand function is $M_t/P_t = (1 + \alpha)^t m$ and as before we assume that transfers are proportional to the monetary base, $G_t = gM_t$. Define the real value of nominal liabilities (normalized to account for growth) as

$$x_t \equiv \frac{L_t}{P_t(1 + \alpha)^t}$$

and express the CB budget constraint as

$$\frac{x_{t-1}}{(1 + \pi_t)(1 + \alpha)} - m + g + \frac{m}{(1 + i_t)} = \frac{x_t}{(1 + i_t)}.$$

The CB loss function from inflation is now

$$C = \sum_{t=0}^{+\infty} \beta^t \frac{{\pi_t}^2}{2}$$

To find the solution with commitment in this simple case we can either solve the sequential problem in which we substitute π_{t+1}^e with π_{t+1} or we can set up a recursive problem which for all $t \geq 1$ has two state variables: the previous period real value of nominal liabilities and a promised inflation rate. In this case the choice variables are real value of nominal liabilities and a promised inflation for t+1. For $t = 0$ the only state variable is the real value of nominal liabilities inherited from $t = -1$ since there is no initial promise to keep from the past. Let's follow the latter approach.

Suppose that at period $t - 1$ the CB made a promised about period t inflation which it has to keep. This promise together with the real value of nominal liabilities, x_{t-1}, are the state variables at t. The CB´s problem at t is to determine the end of period nominal liabilities and make a new promise about future inflation

$$\hat{C}(x, \pi) = \min_{x', \pi'} \left\{ \frac{\pi^2}{2} + \beta \hat{C}(x', \pi') \right\}$$

subject to

$$\frac{x}{(1 + \pi)(1 + \alpha)} - m + g + \frac{m}{(1 + r)(1 + \pi')} = \frac{x'}{(1 + r)(1 + \pi')}.$$

In the initial period $(t = 0)$ there is no promise to keep so the problem is

$$C(x) = \min_{x', \pi, \pi'} \left\{ \frac{\pi^2}{2} + \beta \hat{C}(x', \pi') \right\}$$

subject to the same budget constraint as the previous problem. The continuous time version of the model is a linear quadratic optimization problem so we will solve the discrete time version using a linear quadratic approximation. The details are provided in the appendix together with a description of the sequential problem and its solution.

3) No Commitment.

Now we study the case in which the CB has no technology that allows it to commit to a predetermined course of action. First, we are going to concentrate in the one in which expectations are formed using the current real value of nominal liabilities as the only input. Later we show that there is another equilibrium in which expectations are a function of time only and we discuss how this second equilibrium might be interpreted as an inflation target. Existence of other equilibriums is not discarded. As in the case with commitment we present the continuous and discrete time versions of the model.

a) No commitment in Continuous Time.

We assume that expected inflation is a differentiable function of the real value of nominal liabilities,

$$\pi^e(t) = f\big(x(t)\big).$$

The CB knows this function and takes it as given. The law of motion for x is

$$\dot{x}(t) = \Big(r - \alpha + f\big(x(t)\big) - \pi(t)\Big)(x(t) - m) - (\pi(t) + \alpha - g)m$$

The Hamiltonian is now

$$H = \frac{\pi^2}{2} + \lambda[(r - \alpha + f(x) - \pi)(x - m) - (\pi + \alpha - g)m]$$

The optimality conditions are

$$\pi = \lambda x$$

$$\dot{\lambda} = \lambda\rho - \lambda[r - \alpha + f(x) - \pi + (x - m)f'(x)]$$

together with the law of motion for x and the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)x(t) = 0$$

Let $F(x; f)$ be the solution to the CB´s problem. An equilibrium with rational expectations is defined as a policy function $F(x; f)$ that satisfies

$$F(x; f) = f(x)$$

In general one would have to solve this problem using some trial and error algorithm that converges to the equilibrium. Fortunately since we are working in continuous time we can compute the equilibrium expectations together with the CB´s policy function in one step. We do so next. In equilibrium $f(x) = \pi$ so

$$\dot{\lambda} = \lambda\rho - \lambda\big(r - \alpha + (x - m)f'(x)\big)$$

and

$$\dot{x} = (r - \alpha)x - (r - g)m - \pi m.$$

Since we assumed that $f(x)$ is differentiable we can use the implicit function theorem to write

$$f'(x) = \frac{\dot{\pi}}{\dot{x}}$$

as long as $\dot{x} \neq 0$ which gives us

$$\frac{\dot{\lambda}}{\lambda} = \rho - r + \alpha - (x - m)\frac{\dot{\pi}}{\dot{x}}$$

Combine this equation with the log-derivative with respect to time of first order condition for π

$$\frac{\dot{\pi}}{\pi} = \frac{\dot{\lambda}}{\lambda} + \frac{\dot{x}}{x}$$

to get the system of non linear differential equations given by

$$\dot{\pi} = \frac{(r - \alpha)x - (r - g)m - \pi m}{(r - \alpha)x - (r - g)m - (2m - x)\pi}\left[\rho - (r - g + \pi)\frac{m}{x}\right]\pi$$

and the law of motion for x. The steady state is the same as in the case with commitment

$$\pi^* = 0,$$

$$x^* = \frac{r - g}{r - \alpha}m.$$

If we linearize the system around the steady state we get that in a neighborhood of (x^*, π^*) the system behaves exactly as in the world in which the CB can commit[5].

$$\dot{\pi} = -(r - \alpha - \rho)\pi,$$

$$\dot{x} = (r - \alpha)x - (r - g)m - \pi m.$$

So, close to the steady state, we have

$$\pi(t) = (2(r - \alpha) - \rho)\frac{x(t) - x^*}{m}.$$

The complete solution can be worked out numerically by running the nonlinear system backwards in time. The interesting feature of this solution is that it implies that $\pi(0)$ will tend to infinity as $x(0)$ gets arbitrarily close to $2m$. To see this let's look at the term that appears in the denominator of $\dot{\pi}$. This term is zero if

$$\pi = \frac{(r - \alpha)x - (r - g)m}{2m - x},$$

which implies that the locus $\dot{\pi} = \infty$ has a vertical asymptote at $x = 2m$. If $x^* < 2m$ this vertical asymptote is upward sloping for $x^* < x < 2m$ which implies that the equilibrium path cannot cross it and the initial inflation is forced to diverge to infinity as $x(0)$ tends to $2m$. In this case the solution approximates a discrete swap of the interest bearing liabilities for base money with a burst of inflation. If $x(0) \geq 2m$ then the CB that cannot commit is forced to generate a discrete jump in the initial price level so that it liquates the real value of nominal liabilities.

b) No Commitment in Discrete Time.

As in the continuous time case we study the case in which

$$\pi_{t+1}^e = f(x_{t-1}).$$

The CB´s problem is to solve the following problem taking the function $f(\cdot)$ as given

$$C(x; f) = \min_{x', \pi}\left\{\frac{\pi^2}{2} + \beta C(x'; f)\right\}$$

subject to

$$\frac{x}{(1 + \pi)(1 + \alpha)} - m + g + \frac{m}{(1 + r)\big(1 + f(x)\big)} = \frac{x'}{(1 + r)\big(1 + f(x)\big)}.$$

[5] As long as $(r - \alpha)x - (r - g)m - (2m - x)\pi \neq 0$.

Let

$$\pi = F(x; f)$$

and

$$x' = H(x; f)$$

be the policy functions that solve the previous problem given that expectations are formed using $f(x)$. We define a Markov perfect equilibrium as a pair of functions $F(x; f)$ and $H(x; f)$ which solve the previous problem and satisfy

$$F(H(x; f); f) = f(x).$$

Algorithm for numerical solution:

1. Given $C_0(x)$ and $f_0(x)$ and solve
$$C_1(x) = \min_{x',\pi} \left\{ \frac{\pi^2}{2} + C_0(x') \right\}$$
subject to
$$\frac{x}{(1+\pi)(1+\alpha)} - m + g + \frac{m}{(1+r)\big(1+f(x)\big)} = \frac{x'}{(1+r)\big(1+f(x)\big)}.$$
Let $F_1(x)$ and $H_1(x)$ be the policy functions.
2. Compute
$$f_1(x) = F_1\big(H_1(x)\big).$$
3. If $\max(\|C_1(x) - C_0(x)\|; \|f_1(x) - f_0(x)\|) < \varepsilon$ then stop, otherwise set $C_0(x) = C_1(x)$, $f_0(x) = f_1(x)$ and go back to step 1.

4) Inflation Targeting in Continuous Time.

We now show that under no commitment there is another equilibrium in which expectations are a function of time and $x(0)$ which implies that for $t > 0$ they do not depend on the current state of the economy, $x(t)$. We think that this equilibrium, although it is not sub game perfect, can be interpreted as an economy in which the CB announces an inflation target over time. The target is a self-fulfilling one since if the agents in the economy believe in it then the CB has no incentive to deviate from that target. The equilibrium is not sub game perfect since expectations are fixed at the equilibrium path levels and do not adjust in off equilibrium paths.

Guess that expected inflation is a function of time and $x(0)$ only, i.e. $\pi^e(t) = f\big(t, x(0)\big)$. The CB knows this function and takes it as given. The law of motion for x is

$$\dot{x}(t) = \big(r - \alpha + \pi^e(t) - \pi(t)\big)(x - m) - (\pi(t) + \alpha - g)m$$

The Hamiltonian is now

$$H = \frac{\pi^2}{2} + \lambda[(r - \alpha + \pi^e - \pi)(x - m) - (\pi + \alpha - g)m]$$

The optimality conditions are

$$\pi = \lambda x$$

$$\dot{\lambda} = \lambda\rho - \lambda(r - \alpha + \pi^e - \pi)$$

together with the law of motion for x and the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)x(t) = 0$$

With rational expectations we have that $\pi^e = \pi$ so we get the following system of differential equations:

$$\dot{\pi} = \left[\rho - (r - g + \pi)\frac{m}{x}\right]\pi$$

and

$$\dot{x} = (r - \alpha)(x - m) - (\pi + \alpha - g)m$$

which characterize the solution. The discrete time version of this model is solved in the appendix.

5) Numerical Simulations.

The parameter values for the numerical simulations of the continuous time model are the following

Table 1: Parameter values.				
r	ρ	α	g	m
0.05	0.01	0	0	1

For the discrete time version we compute the solution for different period lengths: one year, one month, one week and one day. The parameter values of the discrete models are set to match the continuous time parameters, i.e.

$$r_{freq} = e^{\frac{r}{freq}} - 1$$

where $freq = \{1{,}12{,}52{,}365\}$ is the model frequency. All results are presented in continuously compounded rates and the CB´s loss function is multiplied by a factor

$$A = \frac{freq^2\left(1 - e^{-\frac{\rho}{freq}}\right)}{\rho}$$

to make comparisons easier across models.

Figure 1 shows the policy function for the continuous time models with and without commitment, and the inflation targeting model. For the models with commitment and with inflation targeting, inflation is almost a linear function of the real value of nominal liabilities. For the particular parameter the initial inflation rate is 9% for the case with commitment and 11.1% for the case with inflation targeting when $x(0) = 2$. These levels of inflation might seem excessive for most developed countries but they well below average inflation for Latin American economies. For the model without commitment inflation tends to infinity as $x(0) \rightarrow 2$.

Figure 1: Policy Function (Continuous Time)

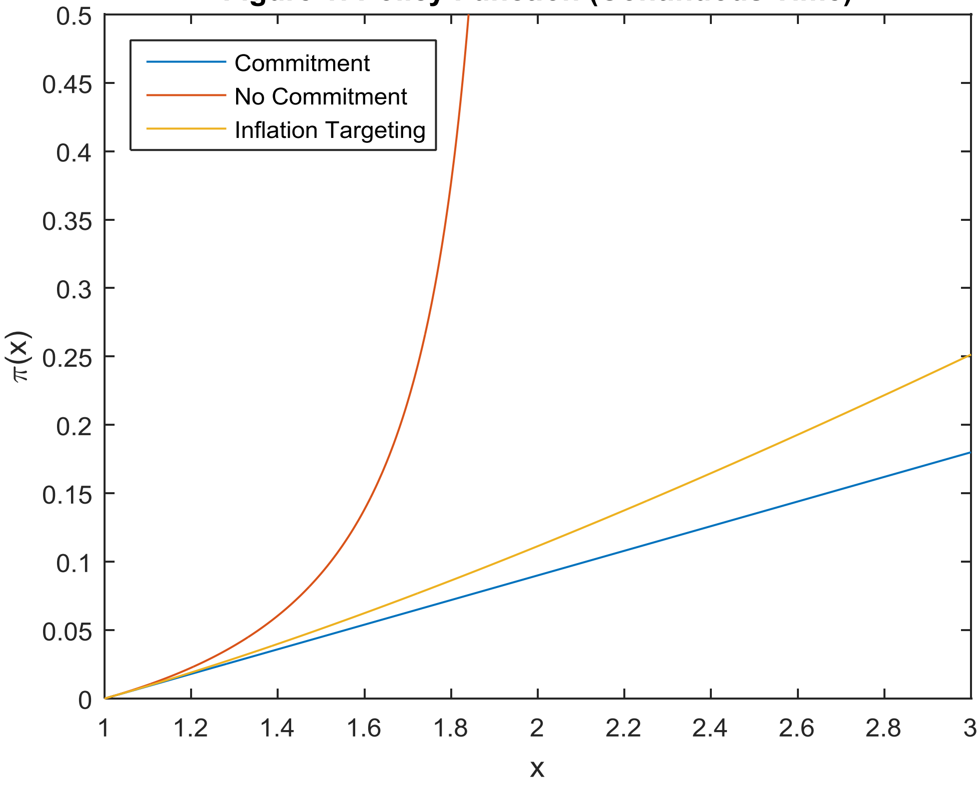
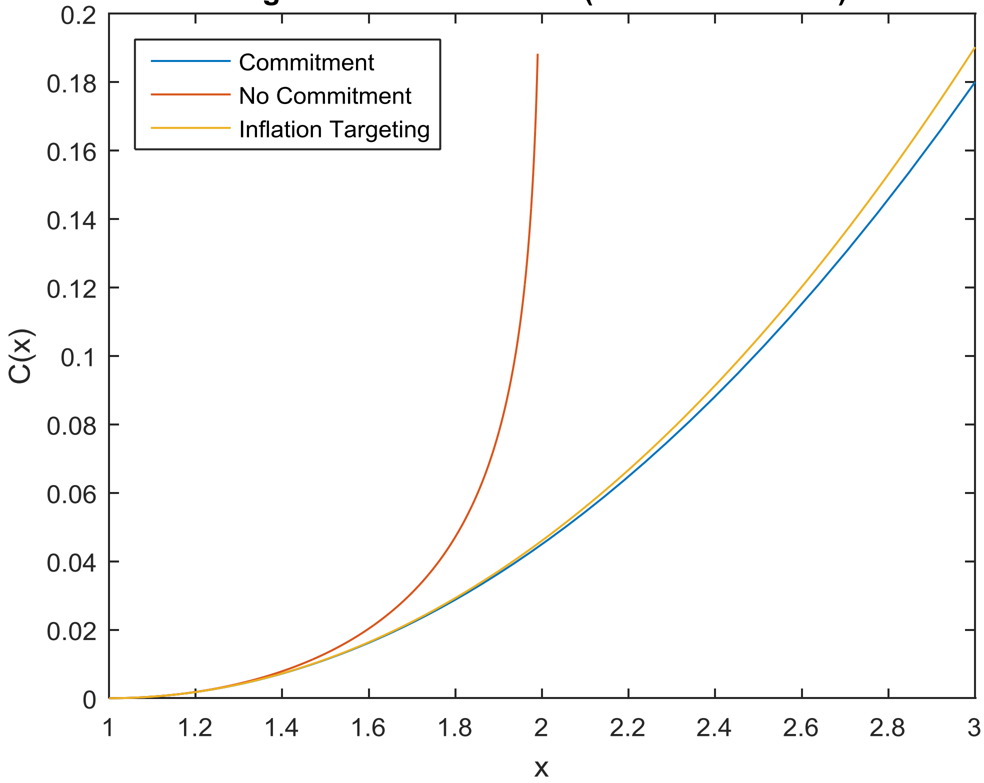


Figure 2 shows the CB loss function for the three continuous time models. For low to medium levels of nominal liabilities the loss for the CB is very similar for all three models, but as the real value of nominal liabilities increase over 150% of the monetary base then the loss without commitment grows very fast and tends to infinity as $x(0) \rightarrow 2$. For the case with commitment and the inflation targeting case the loss for the central bank is very similar even at very high levels of nominal liabilities.

Figure 2: Loss Function (Continuous Time)



We now present the same figures but for the discrete time models with different period lengths. It should be noted that the policy function shown for the model with commitment is valid for $t \geq 1$. For the initial period, $t = 0$, inflation is determined by a different policy function which we show separately in Figure 11. Figures 3 to 6 show the policy functions for yearly, monthly, weekly and daily time lengths, while figures 7 to 10 show the CB´s loss functions.

As expected, higher liabilities lead to higher inflation (with and without commitment) and the discrete time version of the model approximates the continuous time solution for very short time periods (a day or a week). But for longer time periods the solutions are quite different, especially for the case without commitment in which the CB attempts to manipulate expectations through the stock of nominal liabilities. At first we thought that the time period is also the duration of the one period bond that the CB issued this was capturing a debt duration effect but it turns out that this is not the case since we can solve the continuous time version of the model when the CB has access to perpetuities instead of one period bonds and reach the same result.

Figure 3: Policy Function (Discrete Time, Freq = 1)

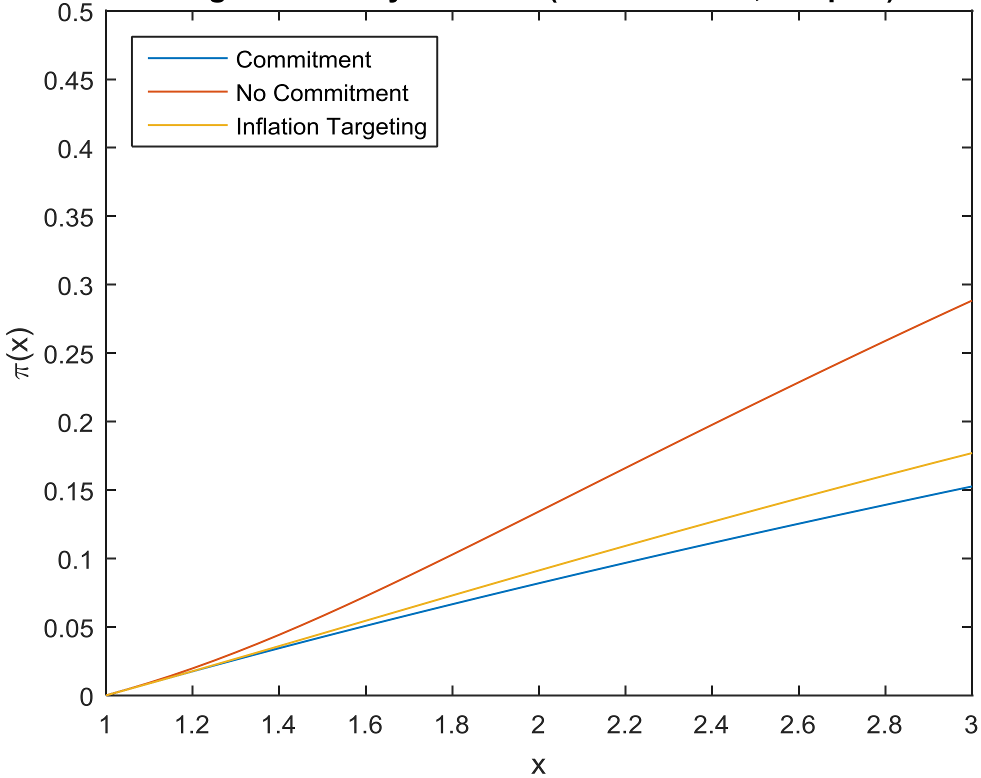


Figure 4: Policy Function (Discrete Time, Freq = 12)

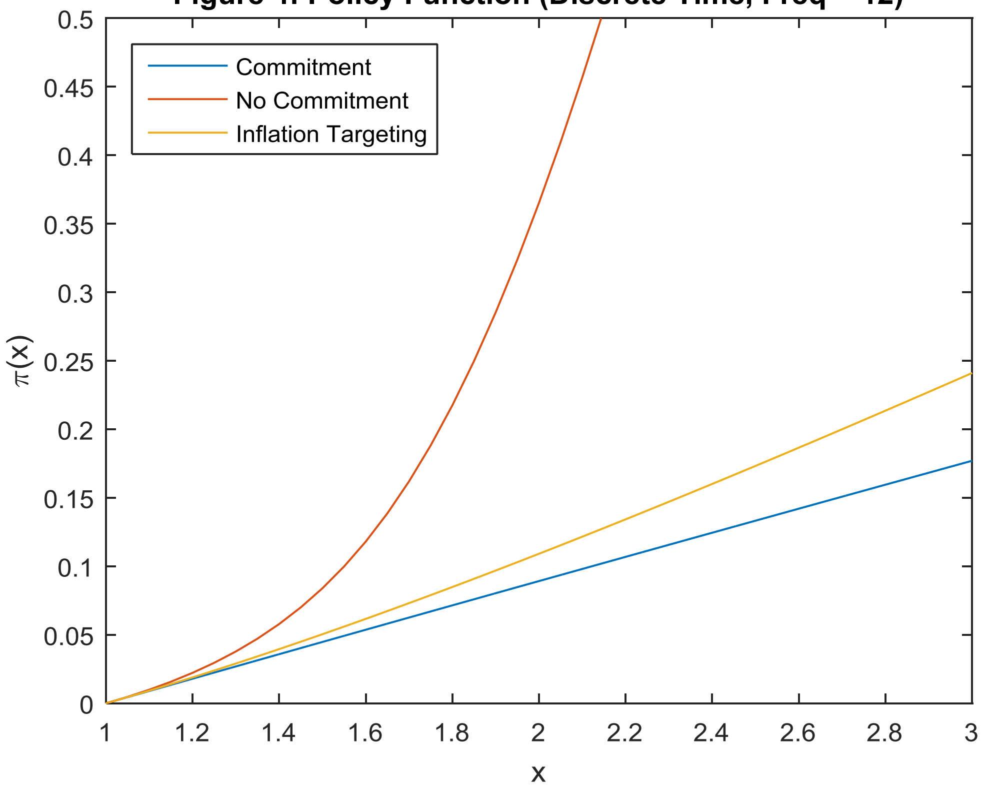


Figure 5: Policy Function (Discrete Time, Freq = 52)

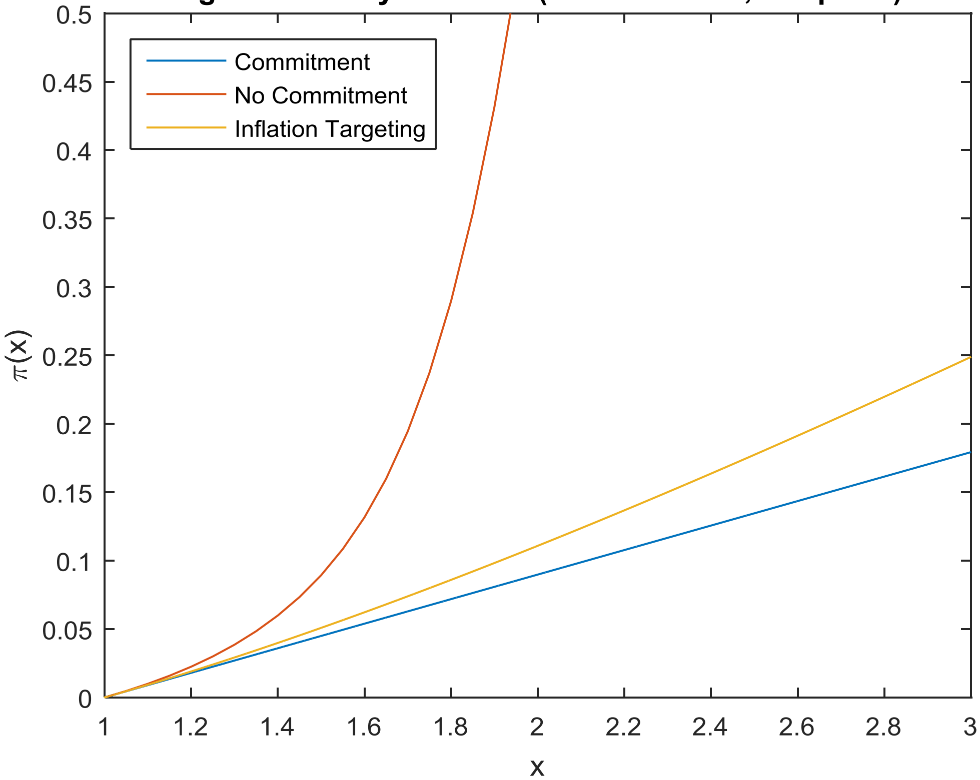


Figure 6: Policy Function (Discrete Time, Freq = 365)

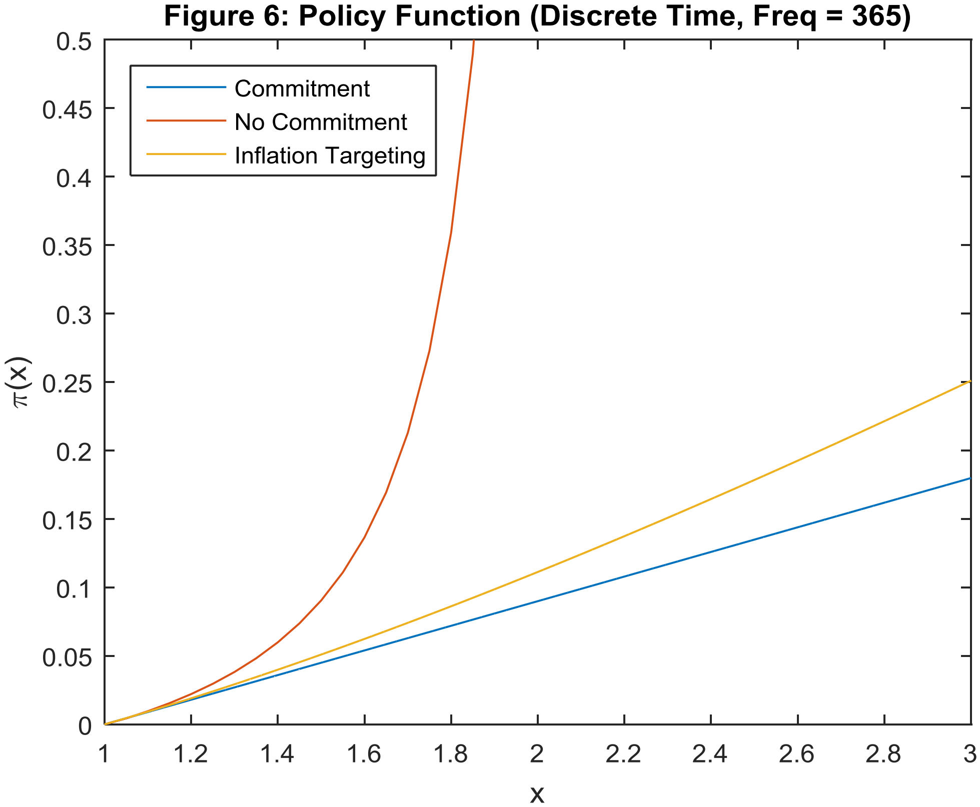


Figure 7: Loss Function (Discrete Time, Freq = 1)

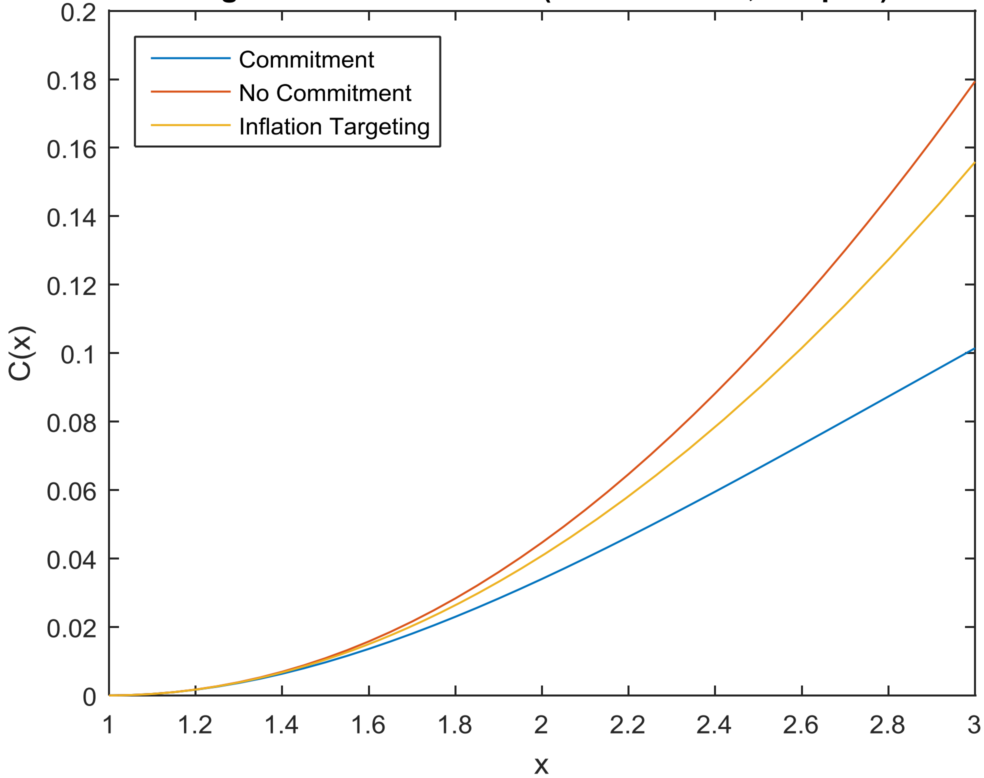


Figure 8: Loss Function (Discrete Time, Freq = 12)

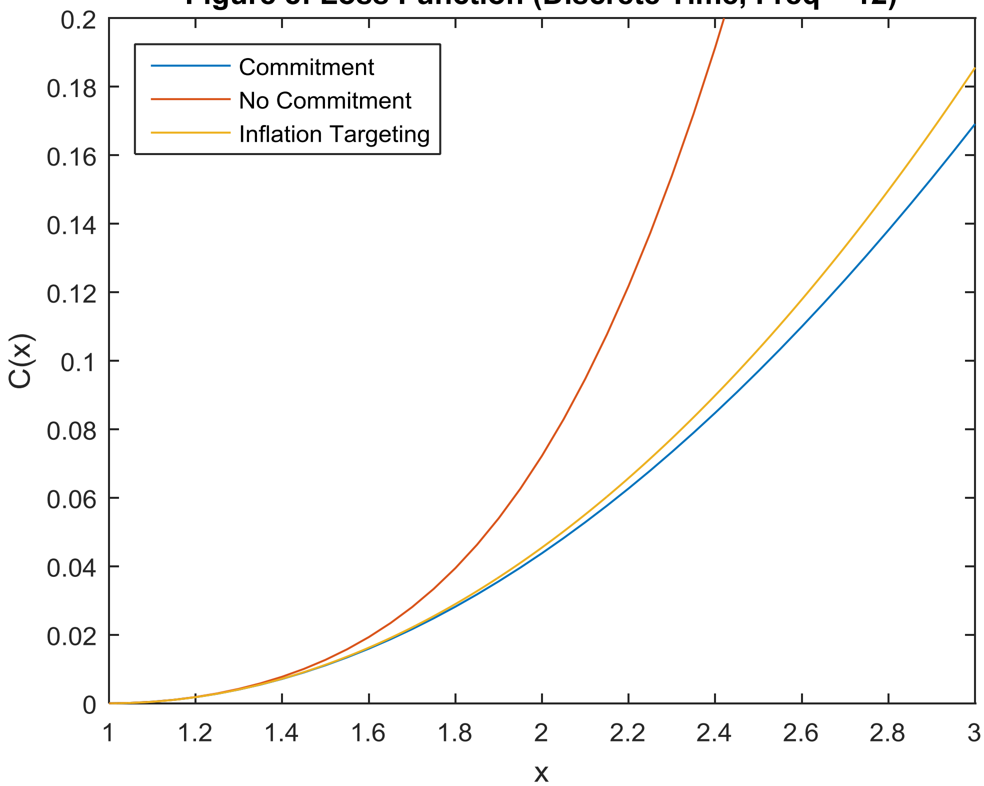


Figure 9: Loss Function (Discrete Time, Freq = 52)

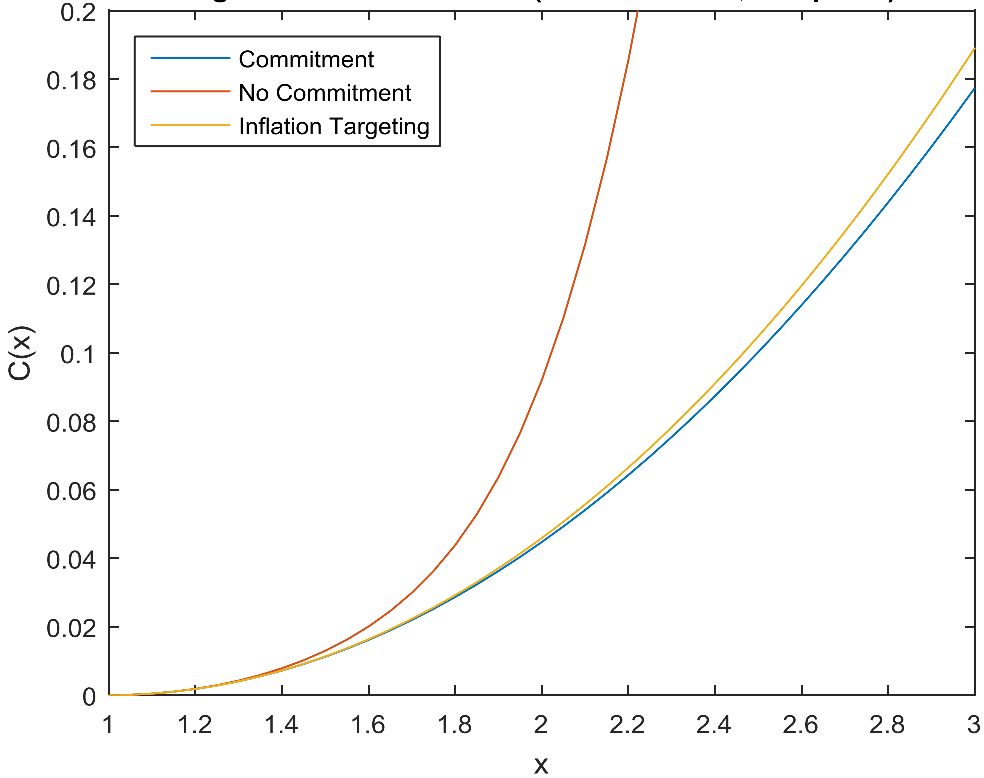


Figure 10: Loss Function (Discrete Time, Freq = 365)

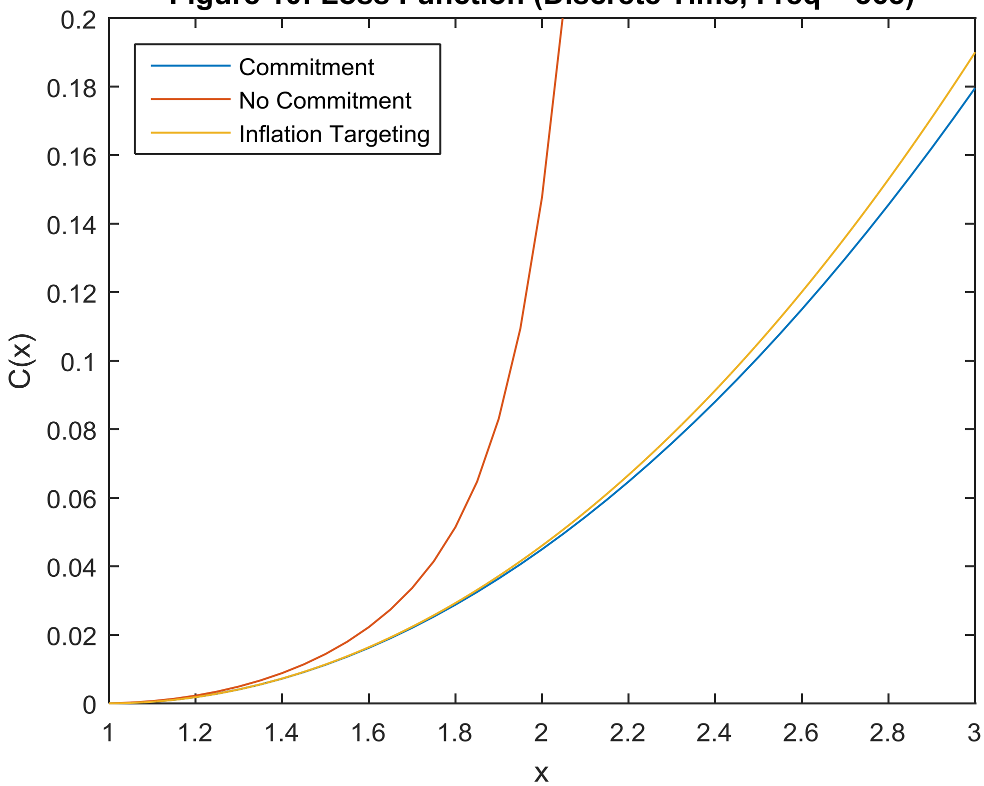
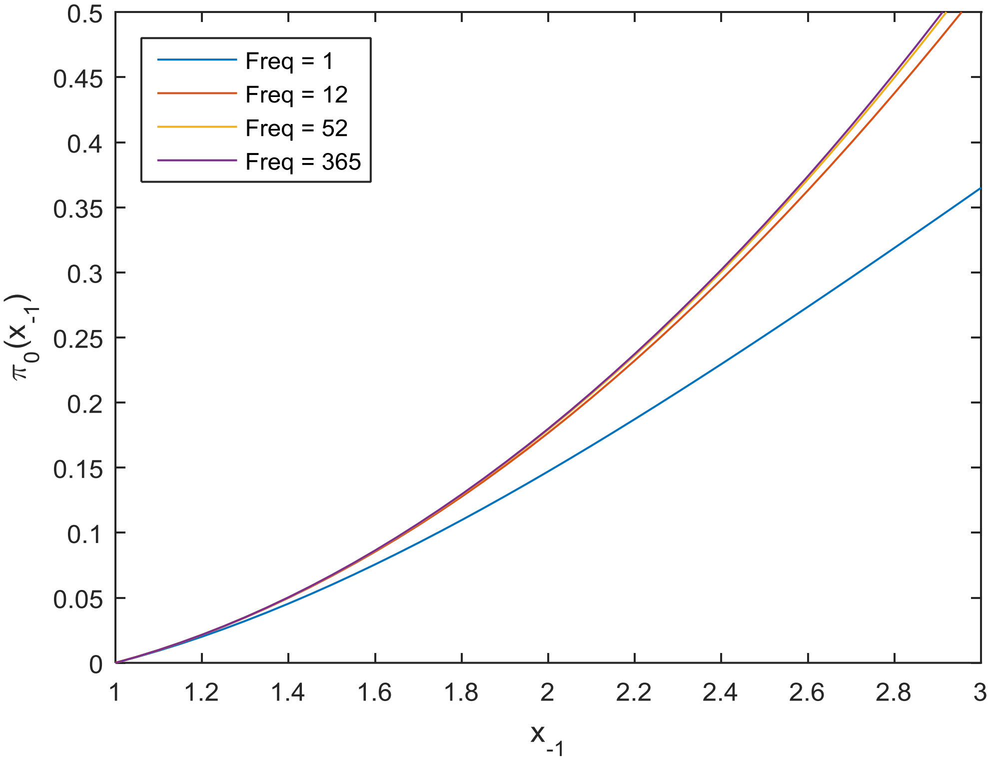


Figure 11 shows the period 0 policy function for the model with commitment. This policy function is different from the policy function for all other periods since at $t = 0$ the CB has no promise to keep.

Figure 11: Period 0 Policy Function with Commitment in Discrete Time



We now present the time paths (price level, inflation and nominal liabilities) for the discrete time model without commitment for different initial levels of nominal liabilities. Figures 12 to 14 show the time paths for an initial level of nominal liabilities $x_{-1} = 1.75$. Figures 15 to 17 and Figures 18 to 20 show the time paths for an initial levels of nominal liabilities $x_{-1} = 2.25$ and $x_{-1} = 3$ respectively.

Figure 12: Inflation Path
No Commitment in Discrete Time (x_{-1}=1.75)

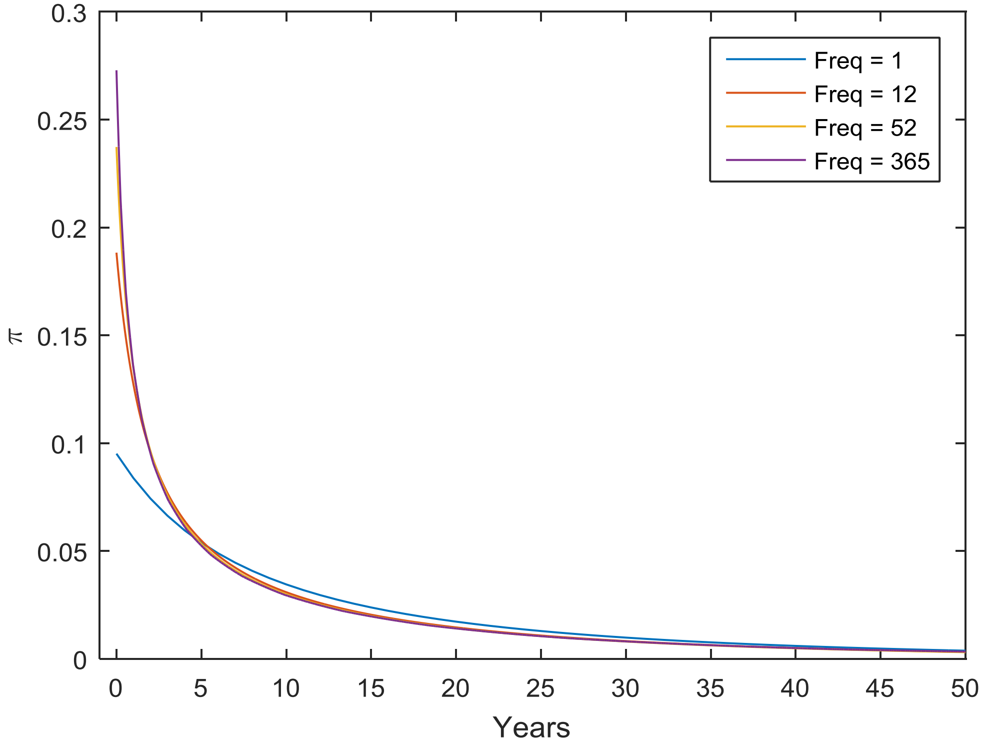


Figure 13: Real value of Nominal Liabilties
No Commitment in Discrete Time (x_{-1}=1.75)

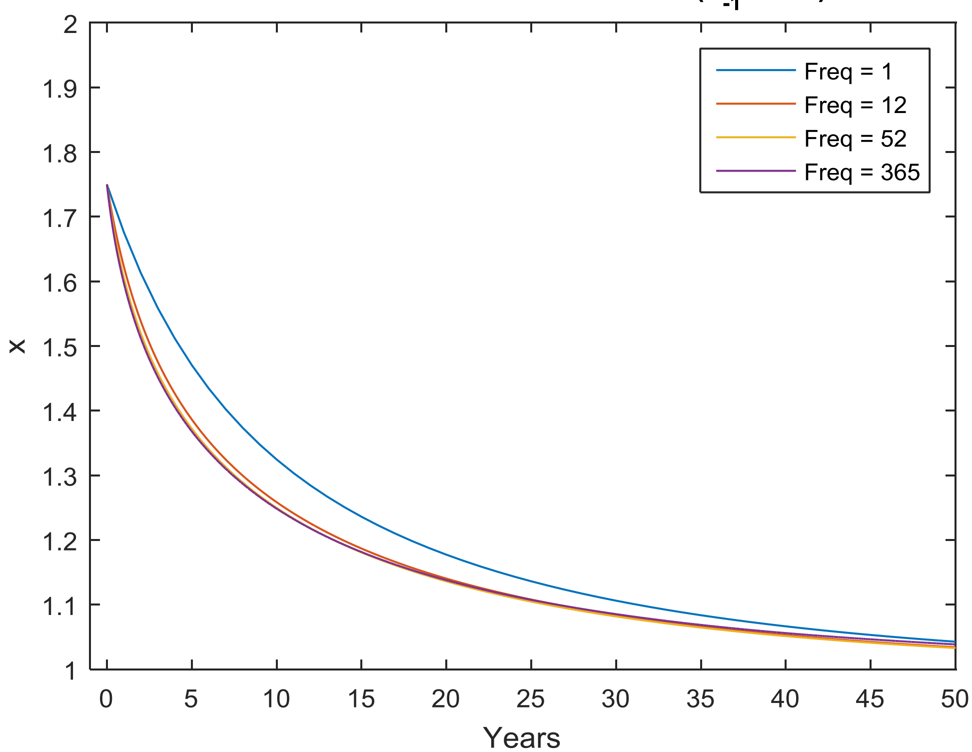


Figure 14: Price Level
No Commitment in Discrete Time (x_{-1}=1.75)

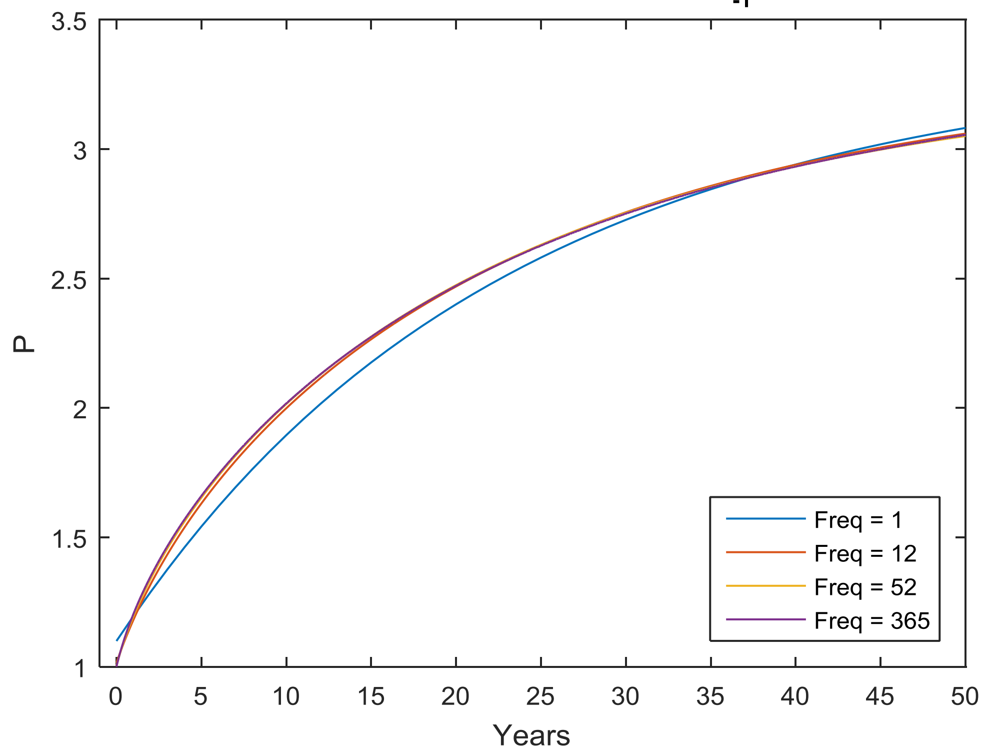


Figure 15: Inflation Path
No Commitment in Discrete Time (x_{-1}=2.25)

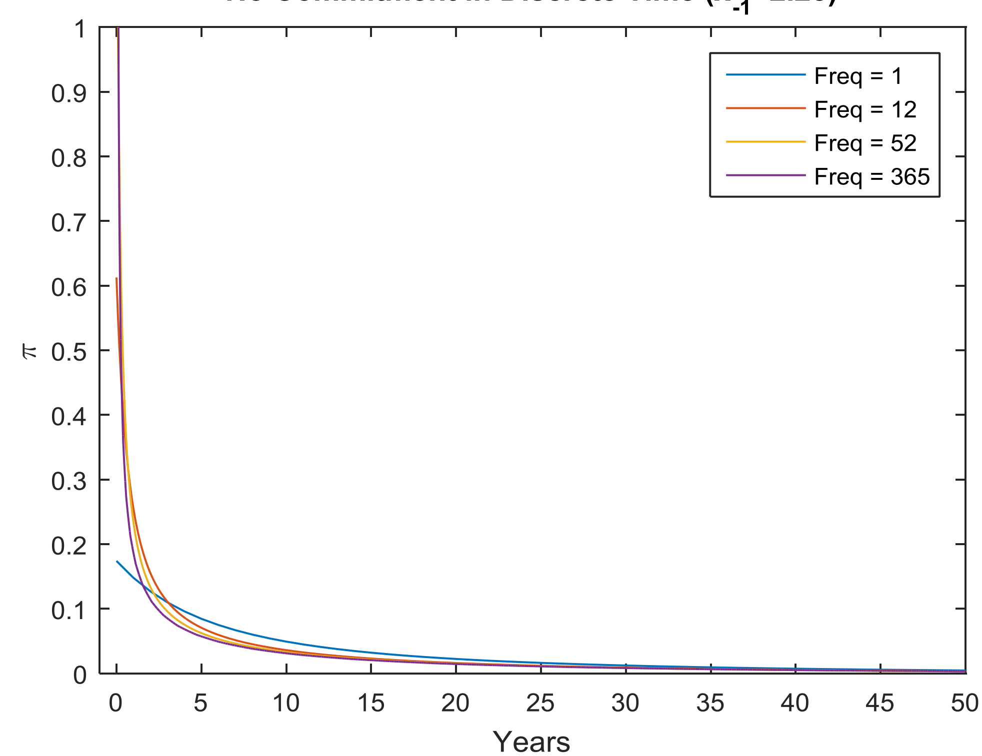


Figure 16: Real value of Nominal Liabilties
No Commitment in Discrete Time (x_{-1}=2.25)

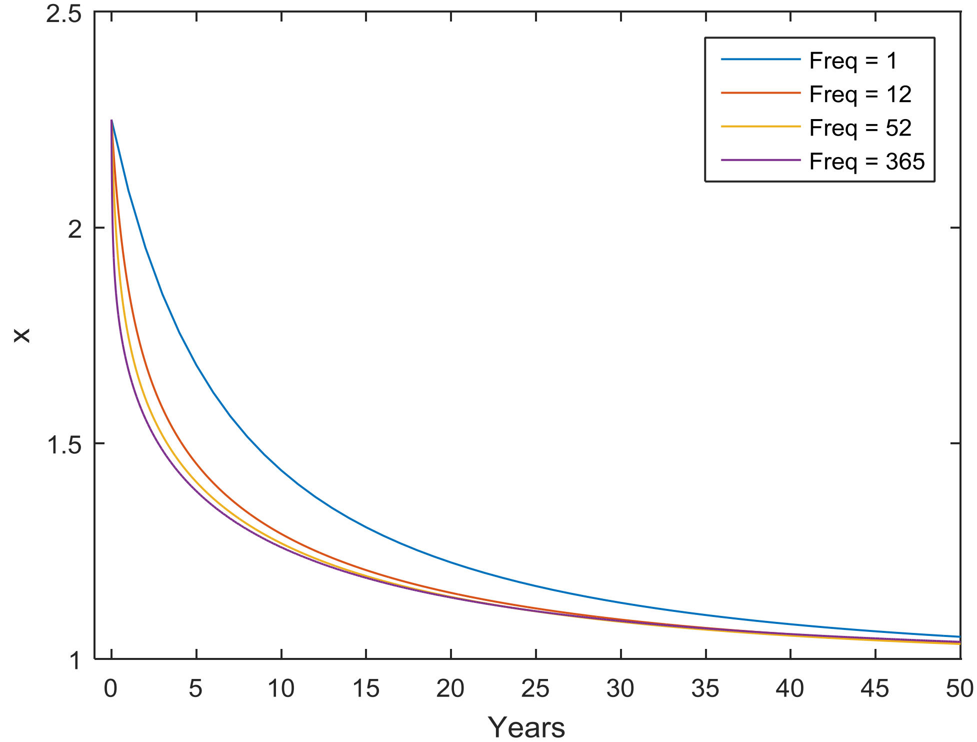


Figure 17: Price Level
No Commitment in Discrete Time (x_{-1}=2.25)

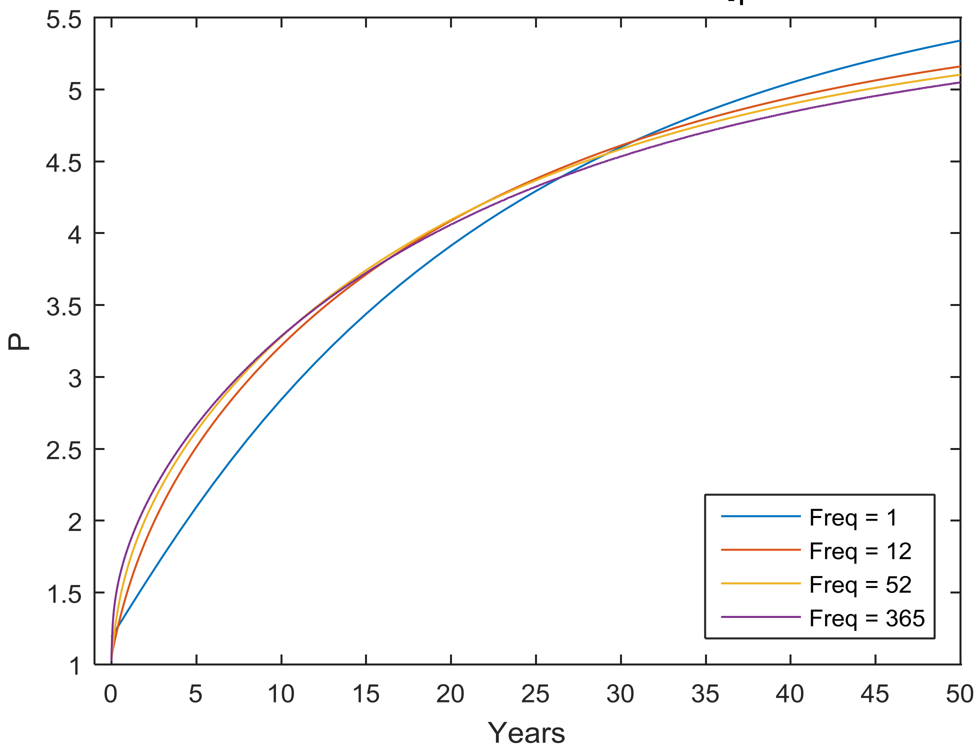


Figure 18: Inflation Path
No Commitment in Discrete Time (x_{-1}=3)

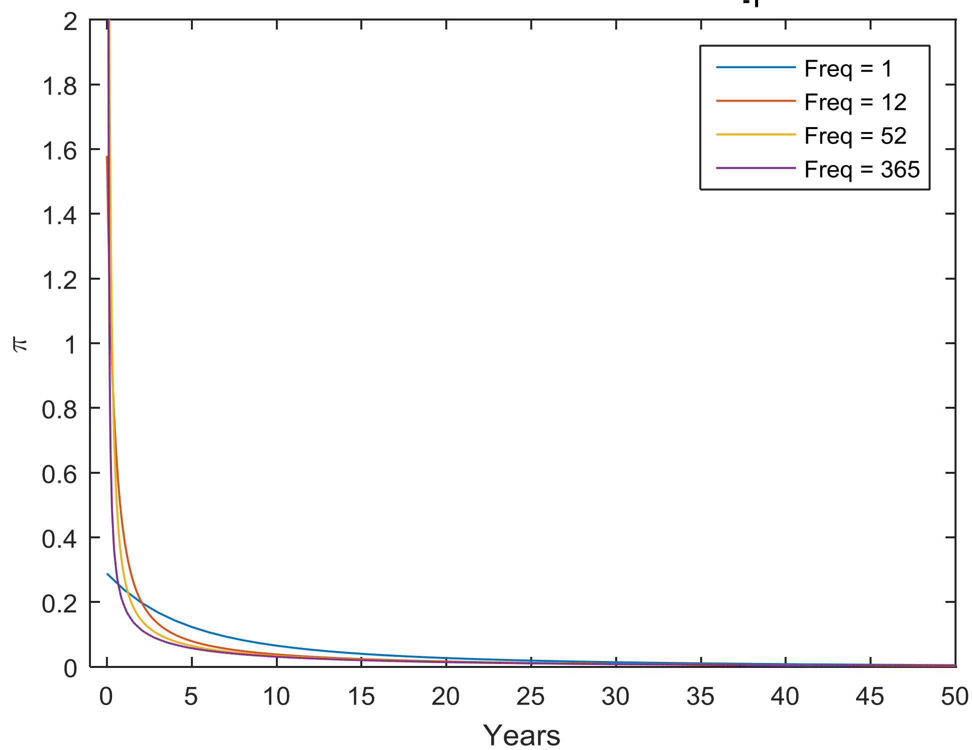


Figure 19: Real value of Nominal Liabilties
No Commitment in Discrete Time (x_{-1}=3)

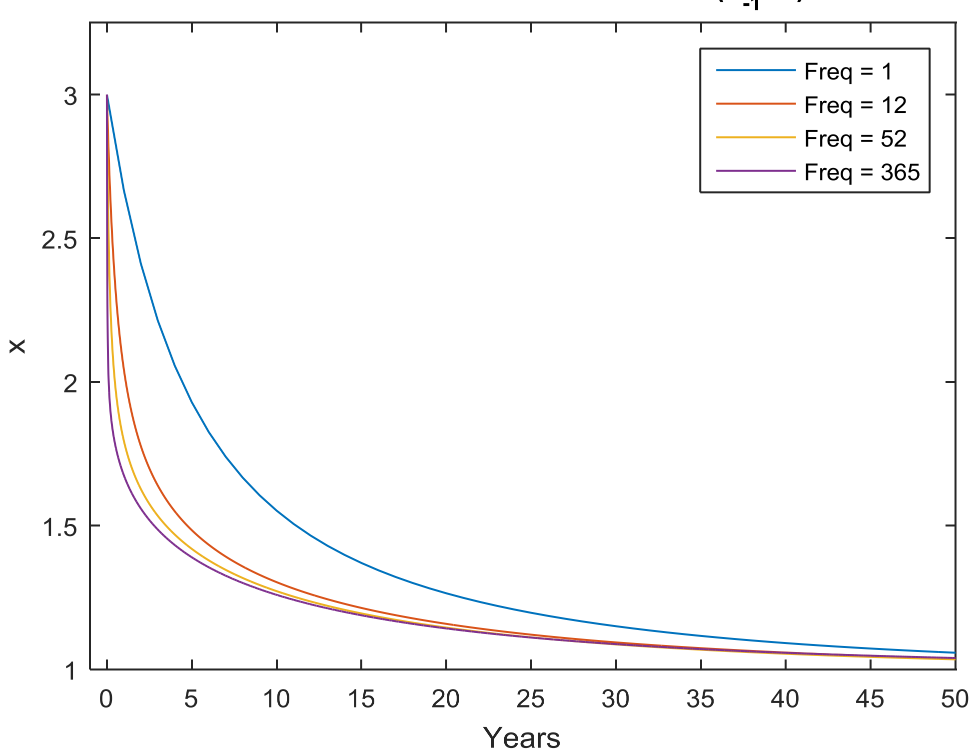
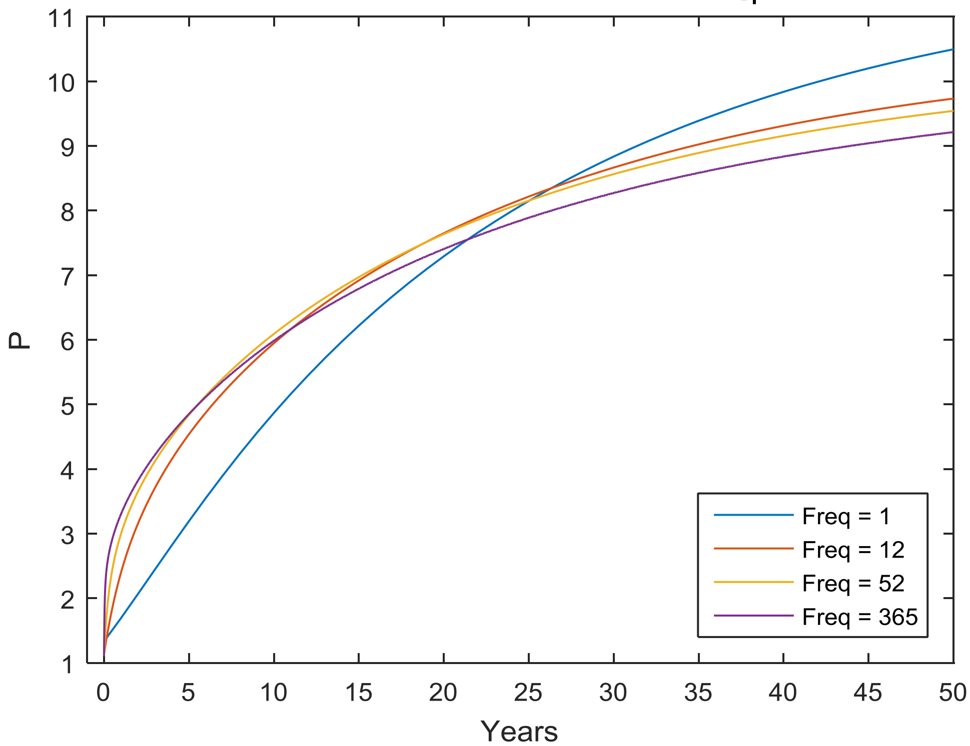


Figure 20: Price Level
No Commitment in Discrete Time (x_{-1}=3)



To summarize the results we present Table 2 that shows the accumulated inflation over the first year for each of the initial levels of nominal liabilities for the model without commitment.

Table 2: First Year Accumulated Inflation (Without Commitment)

x_{-1}	$Freq = 1$	$Freq = 12$	$Freq = 52$	$Freq = 365$
1.75	10%	17%	19%	20%
2.25	19%	50%	69%	82%
3.00	33%	132%	198%	230%

6) Real Debt

We now show that in the continuous time model the solution with commitment is equivalent to an equilibrium in which debt is real[6]. If debt is real the CB´s budget constraint becomes

$$rP(t)d(t) + G(t) = \dot{M}(t) + P(t)\dot{d}(t).$$

where $d(t)$ is the stock of real debt. Total liabilities are given by

[6] This result holds for the case in which money demand does not depend on the nominal interest rate.

$$L(t) = P(t)d(t) + M(t).$$

Substitute the second equation on the first one to obtain.

$$r\big(L(t) - M(t)\big) + G(t) = \dot{M}(t) + P(t)\dot{d}(t)$$

The change in liabilities is now given by

$$\dot{L}(t) = \dot{P}(t)d(t) + P(t)\dot{d}(t) + \dot{M}(t)$$

Use this and the assumption that $G(t) = gM(t)$ to obtain

$$\dot{L}(t) = \big(r + \pi(t)\big)\big(L(t) - M(t)\big) + gM(t)$$

which implies that nominal liabilities increase because the CB has to pay interests on the its debt (with the caveat that now nominal interest payments are not determined by the expected inflation rate but by the actual inflation rate) and transfer resources to the treasury. From here it is straight forward to get

$$\dot{x}(t) = (r - \alpha)(x(t) - m) - (\pi(t) + \alpha - g)m$$

which is the law of motion for the model with commitment discussed in section 2. This implies that if the CB cannot commit then it might be better off by swapping its nominal debt for real debt. This is what Argentina did in December 1989 when demand deposits at commercial banks were exchanged for a 10 year dollar denominated bond call BONEX 89. Although these deposits were a liability of the financial system it should be noted that average reserve requirements (in various forms) exceeded 65%[7] with marginal rates of almost 90%. The CB paid interests on these reserve requirements which were one of the pillars of the quasi-fiscal deficit that fuelled the hyperinflation.

7) Concluding Remarks

We presented a model in which the Central Bank faces nominal liabilities which are in excess of the market´s demand for base money at the initial price level. The CB then faces the decision of when to monetize those nominal liabilities and the classic unpleasant monetarist arithmetic from Sargent and Wallace (1981) kicks in. This implies that the CB can choose to issue debt to sterilize the excess nominal liabilities which results in lower inflation today at the expense of creating more nominal liabilities in the future through interest payments (quasi-fiscal deficit). We compute the optimal solution for the CB in a world in which the institution can commit to a specific policy and find that for reasonable parameter values the resulting inflation is not very high. Adding growth in the money demand makes those numbers smaller. Adding transfers from the CB to the treasury works in the opposite direction. We also solve the model in a world in which the CB cannot

[7] See BCRA 1998 page 22.

commit and agents form their expectations rationally using the real value of nominal liabilities as part of their information set. We show that in the continuous time version of the model the instantaneous inflation rate tends to infinity when the initial nominal liabilities are two times the demand for base money. In the discrete time version this result does not hold anymore although for very short time periods the inflation rate grows very fast with the initial level of nominal liabilities just like in the continuous time model. We also show that in the case that the CB cannot commit there is another rational expectations equilibrium that has much lower inflation. We interpret this equilibrium as one in which the CB announces a time path for inflation. This path is time consistent in the sense that if the agents come to expect the announced inflation then the CB has no incentive to deviate from it. This equilibrium is not sub game perfect since expectations are fixed at the equilibrium path levels and do not adjust in off equilibrium paths.

8) References

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9) Appendix A: Linear Quadratic Approximation for Discrete Time Model with Commitment.

Rewrite the CB´s budget constraint as

$$\frac{x_t}{1+\pi_{t+1}} = \left(\frac{1+r}{1+\alpha}\right)\frac{x_{t-1}}{1+\pi_t} + (1+r)(g-m) + \frac{m}{1+\pi_{t+1}}$$

Define

$$y_t = \begin{bmatrix} 1 \\ \dfrac{x_{t-1}}{1+\pi_t} \\ \dfrac{1}{1+\pi_t} \end{bmatrix}$$

and

$$u_t = \frac{1}{1+\pi_{t+1}} - \frac{1}{1+\pi_t}$$

The law of motion for y_t is the given by

$$\begin{bmatrix} 1 \\ \dfrac{x_t}{1+\pi_{t+1}} \\ \dfrac{1}{1+\pi_{t+1}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ (1+r)(g-m) & \dfrac{1+r}{1+\alpha} & m \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \dfrac{x_{t-1}}{1+\pi_t} \\ \dfrac{1}{1+\pi_t} \end{bmatrix} + \begin{bmatrix} 0 \\ m \\ 1 \end{bmatrix} \left(\frac{1}{1+\pi_{t+1}} - \frac{1}{1+\pi_t}\right)$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ (1+r)(g-m) & \dfrac{1+r}{1+\alpha} & m \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ m \\ 1 \end{bmatrix}$$

or

$$y_{t+1} = Ay_t + Bu_t$$

The loss function can be approximated using a second order Taylor expansion as

$$R(y,u) \cong y'Ry$$

where

$$R = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

Assume that y_0 is given. Then the problem has the form of an optimal linear regulator.

$$C(y_0) = y_0{}'Cy_0 = \min_{\{y_{t+1},u_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t y_t{}'Ry_t$$

$$s.t. \, y_{t+1} = Ay_t + Bu_t \; and \; y_0$$

The Bellman equation representation is:

$$y'Cy = \min_{y^*,u}\{y'Ry + \beta y^*{}'Cy^*\} \; s.t. \, y^* = Ay + Bu$$

which can be solved using standard methods. Finally the solution for y_0 is obtained from the following problem

$$\min_{y_0} y_0{}'Cy_0 \;\; s.t. \, Ey_0 = D$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -x_{-1} \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The Lagrangian is

$$L = y_0{}'Cy_0 + \mu' \, [D - Ey_0]$$

The first order conditions are

$$Cy_0 - E'\mu = 0$$

$$Ey_0 = D$$

$$\begin{bmatrix} C & -E' \\ -E & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ -D \end{bmatrix}$$

and the solution is

$$\begin{bmatrix} y_0 \\ \mu \end{bmatrix} = \begin{bmatrix} C & -E' \\ -E & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -D \end{bmatrix}$$

10) Appendix B: Sequential Formulation of the Discrete Time Model with Commitment.

The Lagrangian for the sequential problem is

$$L = \sum_{t=0}^{+\infty} \beta^t \frac{{\pi_t}^2}{2} + \sum_{t=0}^{+\infty} \lambda_t \left[-\frac{x_t}{1+\pi_{t+1}} + \left(\frac{1+r}{1+\alpha}\right)\frac{x_{t-1}}{1+\pi_t} + (1+r)(g-m) + \frac{m}{1+\pi_{t+1}} \right]$$

The first order conditions are

For π_0

$$\pi_0 = \lambda_0 \left(\frac{1+r}{1+\alpha}\right)\frac{x_{-1}}{(1+\pi_0)^2}$$

for π_t $(t \geq 1)$

$$\beta^t \pi_t + \lambda_{t-1}\frac{x_{t-1}-m}{(1+\pi_t)^2} = \lambda_t \left(\frac{1+r}{1+\alpha}\right)\frac{x_{t-1}}{(1+\pi_t)^2}$$

for x_t $(t \geq 0)$

$$\frac{\lambda_t}{1+\pi_{t+1}} = \left(\frac{1+r}{1+\alpha}\right)\frac{\lambda_{t+1}}{1+\pi_{t+1}}$$

And the law of motion for x_t $(t \geq 0)$

$$\frac{x_t}{1+\pi_{t+1}} = \left(\frac{1+r}{1+\alpha}\right)\frac{x_{t-1}}{1+\pi_t} + (1+r)(g-m) + \frac{m}{1+\pi_{t+1}}$$

Evaluate the first order condition for x_s at $s = t-1$

$$\lambda_{t-1} = \left(\frac{1+r}{1+\alpha}\right)\lambda_t$$

And substitute in the first order condition for π_t

$$\beta^t \pi_t (1+\pi_t)^2 = \lambda_t \left(\frac{1+r}{1+\alpha}\right) m$$

What we get is for $t \geq 1$

$$\pi_t (1+\pi_t)^2 = \frac{\pi_0 (1+\pi_0)^2}{x_{-1}} \left(\frac{1+\alpha}{(1+r)\beta}\right)^t m$$

The value of π_0 has to be chosen such that x_t does not explode and it converges to the steady state.

Commitment policy function for $t \geq 1$

$$\beta \pi_{t+1}(1+\pi_{t+1})^2 \frac{1+r}{1+\alpha} = \pi_t(1+\pi_t)^2$$

$$\frac{x_t}{1+\pi_{t+1}} = \left(\frac{1+r}{1+\alpha}\right)\frac{x_{t-1}}{1+\pi_t} + (1+r)(g-m) + \frac{m}{1+\pi_{t+1}}$$

11) Appendix C: Sequential Formulation of the Discrete Time Model without Commitment when $\pi_{t+1}^e = f(t, x_{-1})$

The Lagrangian for the sequential problem is

$$L = \sum_{t=0}^{+\infty} \beta^t \frac{{\pi_t}^2}{2} + \sum_{t=0}^{+\infty} \lambda_t \left[-\frac{x_t}{1+\pi_{t+1}^e} + \left(\frac{1+r}{1+\alpha}\right)\frac{x_{t-1}}{1+\pi_t} + (1+r)(g-m) + \frac{m}{1+\pi_{t+1}^e} \right]$$

The first order conditions are

For π_t $(t \geq 0)$

$$\beta^t \pi_t = \lambda_t \left(\frac{1+r}{1+\alpha}\right)\frac{x_{t-1}}{(1+\pi_t)^2}$$

for x_t $(t \geq 0)$

$$\frac{\lambda_t}{1+\pi_{t+1}^e} = \left(\frac{1+r}{1+\alpha}\right)\frac{\lambda_{t+1}}{1+\pi_{t+1}}$$

And the law of motion for x_t $(t \geq 0)$

$$\frac{x_t}{1+\pi_{t+1}^e} = \left(\frac{1+r}{1+\alpha}\right)\frac{x_{t-1}}{1+\pi_t} + (1+r)(g-m) + \frac{m}{1+\pi_{t+1}^e}$$

Under rational expectations $\pi_{t+1}^e = \pi_{t+1}$ so we get

$$\beta^t \pi_t (1+\pi_t)^2 = \lambda_t \left(\frac{1+r}{1+\alpha}\right) x_{t-1}$$

which is the same as in the case with commitment with the crucial difference that the last term multiplying on the right hand side of the equation in this case is x_{t-1} instead of m. The law of motion for λ_t is the same as before

$$\lambda_t = \left(\frac{1+\alpha}{1+r}\right)\lambda_{t-1}$$

To solve the whole path forward:

1. Given x_{t-1} and π_t: x_t and π_{t+1} solve:

$$\frac{\beta \pi_{t+1}(1+\pi_{t+1})^2}{\pi_t (1+\pi_t)^2} = \left(\frac{1+\alpha}{1+r}\right)\frac{x_t}{x_{t-1}}$$

$$\frac{x_t}{1+\pi_{t+1}} = \left(\frac{1+r}{1+\alpha}\right)\frac{x_{t-1}}{1+\pi_t} + (1+r)(g-m) + \frac{m}{1+\pi_{t+1}}$$

2. Start with x_{-1} and guess an initial value for π_0. Use the previous condition to compute the whole path.
3. Try different values of π_0.until you find one that converges to the steady state

Or backwards:

1. Start close to the steady state and solve backwards. Given x_t and π_{t+1}
$$\frac{\beta\pi_{t+1}(1+\pi_{t+1})^2}{\pi_t(1+\pi_t)^2} = \left(\frac{1+\alpha}{1+r}\right)\frac{x_t}{x_{t-1}}$$
$$\frac{x_t}{1+\pi_{t+1}} = \left(\frac{1+r}{1+\alpha}\right)\frac{x_{t-1}}{1+\pi_t} + (1+r)(g-m) + \frac{m}{1+\pi_{t+1}}$$
$$\left[\frac{x_t - m}{1+\pi_{t+1}} - (1+r)(g-m)\right]\left(\frac{1+\alpha}{1+r}\right) = \frac{x_{t-1}}{1+\pi_t}$$
$$\frac{x_{t-1}}{1+\pi_t} = \left[\frac{1+\alpha}{(1+r)\beta}\right]\frac{x_t}{\pi_{t+1}(1+\pi_{t+1})^2}\pi_t(1+\pi_t)$$
2. Solve quadratic equation for π_t
$$\left[\frac{x_t - m}{1+\pi_{t+1}} - (1+r)(g-m)\right]\left(\frac{1+\alpha}{1+r}\right) = \left[\frac{1+\alpha}{(1+r)\beta}\right]\frac{x_t}{\pi_{t+1}(1+\pi_{t+1})^2}\pi_t(1+\pi_t)$$
3. Then compute x_{t-1}. Go back to 1.

12) Appendix D: Perpetuity Debt

At time t the CB has nominal liabilities $L(t)$ which consist of nominal debt, $D(t)$, the monetary base $M(t)$,

$$L(t) = q(t)D(t) + M(t).$$

Debt is assumed to be in the form of a perpetuity that pays a constant flow of $1 each period forever. The period t price of one of these bonds is $q(t)$ which satisfies

$$q(t) = \int_t^{+\infty} e^{-\int_t^s i(v)dv} ds$$

where $i(t)$ is the nominal interest rate. Each period the CB has to pay interests on the debt. These expenses are financed by printing new money or issuing new debt,

$$D(t) = \dot{M}(t) + q(t)\dot{D}(t).$$

We assume that the money demand is perfectly inelastic with respect to the nominal interest rate, i.e.

$$\frac{M(t)}{P(t)} = m.$$

where $P(t)$ is the price level at t. The nominal interest rate satisfies the Fisher equation

$$i(t) = r + \pi^e(t).$$

Define the real value of nominal liabilities as

$$x(t) \equiv \frac{L(t)}{P(t)}.$$

$$\frac{\dot{L}}{L}\frac{L}{P} = (1 + \dot{q})\frac{D}{P}$$

$$\left(\frac{\dot{x}}{x} + \pi\right)x = \frac{(1 + \dot{q})}{q}\frac{L - M}{P}$$

$$\dot{x} = \left(\frac{1 + \dot{q}}{q} - \pi\right)(x - m) - \pi m$$

$$q(t) = \int_t^{+\infty} \exp\left[-\int_t^s \left(r + \pi^e(v)\right)dv\right] ds$$

$$\dot{q}(t) = \int_t^{+\infty} \exp\left[-\int_t^s \big(r + \pi^e(v)\big)dv\right]\big(r + \pi^e(t)\big)ds - 1$$

$$\dot{q}(t) = \big(r + \pi^e(t)\big)q(t) - 1$$

$$1 + \dot{q}(t) = \big(r + \pi^e(t)\big)q(t)$$

With the information presented we can derive the law of motion for x as

$$\dot{x}(t) = \big(r + \pi^e(t) - \pi(t)\big)(x(t) - m) - \pi(t)m,$$

which is the same as before when we had one period debt.