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Germán Coloma

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The Penalty-Kick Game under Incomplete Information

Germán Coloma*

Abstract

This paper presents a model of the penalty-kick game between a soccer goalkeeper and a kicker, in which there is uncertainty about the kicker's type (and there are two possible types of kicker). To find a solution for this game we use the concept of Bayesian equilibrium, and we find that, typically, one of the kicker's types will play a mixed strategy while the other type will choose a pure strategy (or, sometimes, a "restricted mixed strategy"). The model has a simpler version in which the players can only choose between two strategies (right and left), and a more complex version in which they can also choose a third strategy (the center of the goal). Comparing the incomplete-information Bayesian equilibria with the corresponding complete-information Nash equilibria, we find that in all cases the expected scoring probability increases (so that, on average, the goalkeeper is worse off under incomplete information). The three-strategy model is also useful to explain why it could be optimal for a goalkeeper never to choose the center of the goal (although at the same time there were some kickers who always chose to shoot to the center).

JEL Classification: C72 (non-cooperative games), L83 (sports).

Keywords: soccer penalty kicks, mixed strategies, Bayesian equilibrium, incomplete information.

1. Introduction

The penalty-kick game, in which a soccer goalkeeper and a kicker face each other, has become an important example in the applied game-theory literature to analyze mixed-strategy Nash equilibria. The reason of this importance probably has to do with the fact that it is a game whose solution generates a clear theoretical prediction and, at the same time, it is relatively easy to gather data about actual outcomes of the game. Besides, this is a game in which it is not necessary to perform laboratory experiments, since penalty-kick situations in soccer matches are frequent, and soccer players are usually trained to shoot and to save penalty kicks. Moreover, there exist relatively large records of the different details involved in many actual penalty kicks shot at various soccer leagues (e.g., if they were scored or not, the side chosen by the goalkeeper and the kicker, the identity of the goalkeeper and the kicker, the situation of the match at the moment of the shot, etc.), and this helps to control for several factors that may influence the result of the game.

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* CEMA University; Av. Córdoba 374, Buenos Aires, C1054AAP, Argentina. Telephone: (54-11)6314-3000; E-mail: gcoloma@cema.edu.ar. The views and opinions expressed in this publication are those of the author and are not necessarily those of CEMA University. I thank Nicolás Caputo, Timothy Groseclose, Hugo Hopenhayn, John Riley, Connan Snider and Jorge Streb for their useful comments. I also thank the participants of two seminars held at CEMA University and at the University of California, Los Angeles.

All the game-theoretic literature that we know about penalty kicks analyzes this situation as a game of complete information, i.e., as a game in which the two players know the characteristics of each other, and hence they know the expected payoffs that they will receive in the different strategy profiles of the game. There is a good reason for this assumption, which is the idea that goalkeepers and kickers in a professional soccer league are usually well-known players whose main characteristics are recognized by their opponents, and those characteristics are precisely the ones that define the parameters which establish the expected payoff of the penalty-kick game. Complete-information games, moreover, are also easier to solve and, perhaps more importantly, are easier to test empirically. This ease is probably the best explanation for the success of the penalty-kick game as a prominent example in the game-theoretic literature.

Not all soccer penalty kicks, however, are shot in situations in which it is reasonable to assume complete information. In many cases, especially in amateur matches and in matches between teams that belong to different leagues, it is possible that players do not know each other and, therefore, are uncertain about several important characteristics that may influence the outcome of the game. It is also possible that the kicker (and, less usually, the goalkeeper) is a player who is not the "typical choice" in his team, because he is out of the match or because the team has decided to change him due to a poor past performance. Moreover, as penalty kicks are sometimes used as tie-breakers in some tournaments, and this requires that several players from each team shoot penalty kicks, it is possible that some of the designated kickers do not usually shoot penalty kicks in professional matches. This may generate a situation in which the goalkeeper is uncertain about some of the kicker's relevant characteristics, changing the game into one with incomplete information.

When we have to analyze a game with incomplete information, the main solution concept for games with complete information (i.e., Nash equilibrium) is usually unavailable. Since the seminal contribution by Harsanyi (1967), however, we have an alternative concept to apply in these cases, which is the so-called "Bayesian equilibrium". This equilibrium relies on the idea that, under incomplete information, players typically have data about the probabilities of their opponents' characteristics, and this allows them to figure out which are the different "types" of opponents that they may face and the probability associated to each type. With that information we can build an equilibrium in which each player's type plays his best response to their opponents' strategies, taking into account the probability of facing each opponent's type.

In this paper we will develop a model of a penalty-kick game in which there is a single type of goalkeeper and two types of kickers. This is consistent with several results that appear in the literature, and especially with the empirical observation of Chiappori, Levitt and Groseclose (2002) that professional goalkeepers are basically homogeneous in their characteristics as penalty-kick savers, and that the main variation that we observe comes from the kickers' side. In their contribution, for example, they find that the complete-information penalty-kick game can have two different classes of Nash equilibrium, depending on the scoring probability associated to shooting to the center of the goal. If that scoring probability is relatively low, then the equilibrium is what they call a "restricted-randomization equilibrium", in which both the goalkeeper and the kicker randomize between left and right, but they never choose the center. If conversely, the scoring probability of shooting to the center of the goal (when the goalkeeper chooses one of his sides) is relatively large, then we find a "general-randomization equilibrium" (in which both the goalkeeper and the kicker randomize among left, right and center).

In a recent paper by Jabbour and Minquet (2009), the authors allow for an additional strategy dimension which is the height of the kicker's shot, and assume that shooting high to one of the sides (left or right) assures the kicker a certain scoring probability (because the goalkeeper cannot save the shot, and his only hope is that the kicker shoots outside the goal). In this case we also have two classes of Nash equilibrium, which depend on the scoring probability of shooting high: if this probability is relatively small, then the kicker will randomize between shooting right-low or left-low (but he will never shoot high); if it is large, then the kicker will strictly prefer to shoot high, and he will always choose his "natural side" (i.e., the goalkeeper's right, if the kicker is right-footed, or the goalkeeper's left, if the kicker is left-footed).

Different types of kickers also change the Nash equilibrium in the simplest versions of the penalty-kick game. In Palacios-Huerta (2003), for example, both the goalkeeper and the kicker choose between two strategies (left and right), but the equilibrium mixed strategies are functions of the scoring probabilities of the four possible strategy profiles. Changing one of these parameters, therefore, changes the equilibrium; and playing a strategy that is an equilibrium one for a certain set of parameters when the set of parameters is different, consequently, implies that the other player's best response is a pure strategy and not a mixed one. This last situation, of course, is never a Nash equilibrium in a complete information setting, but it may well be part of a Bayesian equilibrium if we assume incomplete information about the kicker's characteristics.

In the next section of this paper we will present a simple model in which we will allow for uncertainty about one of the four parameters that define the scoring probabilities of the 2x2 version of the penalty-kick game. This change, together with the inclusion of an additional parameter that defines the probability distribution of the kicker's types, will generate a new game in which the kicker plays knowing the goalkeeper's characteristics but the goalkeeper plays against an uncertain opponent (who may be of two different types). We will also compare the solution of this game with their complete-information counterparts, i.e., with the Nash equilibria of the games in which the goalkeeper alternatively faces each of the kicker's types, knowing who his opponent is.

In the third section of the paper, we will develop another model in which both the goalkeeper and the kicker can choose among three different strategies (right, left and center), and we will also compare the equilibria of the game under complete and incomplete information. Following that, the fourth section will include a numerical example of the proposed models, and the fifth section will develop an empirical illustration based on data from previous studies about the penalty-kick game. Finally, the sixth section will be devoted to the conclusions of the whole paper and to some final remarks.

2. A model with two strategies

2.1. Complete information

Following the notation that appears in Coloma (2007), we will build a game in which the kicker has to choose between his natural side (the goalkeeper's right, if the kicker is right-footed, or the goalkeeper's left, if the kicker is left-footed) and his opposite side (the goalkeeper's left, if the kicker is right-footed, or the goalkeeper's right, if the kicker is left-footed). Similarly, the goalkeeper has to choose between the kicker's natural side (NS) and the kicker's opposite side (OS). The probability of scoring if both the kicker and the goalkeeper choose NS is P_N, while the probability of scoring if both the kicker and the goalkeeper choose OS is P_O. If the kicker chooses NS but the goalkeeper chooses OS, then the scoring probability is π_N, while the scoring probability in the case that the kicker chooses OS and the goalkeeper chooses NS is π_O. As this is a constant-sum game in which the kicker wins if he scores and the goalkeeper wins if the kicker does not score, then the kicker's expected payoff can be associated to the scoring probability and the goalkeeper's expected payoff can be associated to the complement of that probability. As it is a simultaneous game, then the kicker's strategy space consists of two strategies (NS and OS) and the goalkeeper strategy space also consists of two strategies (NS and OS).

Both the theoretical and the empirical literature agree that the scoring probabilities of the penalty-kick game have to be defined so that "$\pi_N \geq \pi_O > P_N > P_O$", and these conditions guarantee that the Nash equilibrium of the complete-information version of the game is a mixed-strategy one, in which the kicker chooses NS with a certain probability *n* (and chooses OS with probability *1-n*), and the goalkeeper chooses NS with a certain probability *v* (and chooses OS with probability *1-v*). Moreover, the fact that "$P_N > P_O$" makes that, in equilibrium, both *n* and *v* are greater than ½. Besides, if π_N is strictly greater than π_O, then it will also hold that, in equilibrium, "$v > n$", while if it holds that "$\pi_N = \pi_O$", then equilibrium implies that "$n = v$"[1].

One of the easiest ways of building a penalty-kick game with incomplete information is to assume that there is a single type of goalkeeper and two types of kickers. We will assume that kicker 1 is someone for whom "$\pi_N > \pi_O$", while kicker 2 is someone for whom "$\pi_N = \pi_O$". To simplify matters even further, we will assume that the values of "P_N" and "P_O" are the same for both types of kickers, and that "π_O" is also the same for both types. Then our model depends on the standard four parameters (π_N, π_O, P_N and P_O), plus an additional parameter θ that represents the probability that the goalkeeper faces kicker 1 (while the probability of facing kicker 2 is $1\text{-}\theta$). The complete probability matrix is the one that appears on table 1.

Table 1: Scoring-probability matrix for the two-strategy game

		Goalkeeper	
		NS	OS
Kicker 1 (Prob θ)	NS	P_N	π_N
	OS	π_O	P_O
Kicker 2 (Prob 1-θ)	NS	P_N	π_O
	OS	π_O	P_O

If we first consider the case in which "$\pi_N > \pi_O$" (game 1), then the complete-information Nash equilibrium implies that both the goalkeeper and the kicker are indifferent between choosing NS and OS. For this to happen, it should hold that:

$$P_N \cdot n_1 + \pi_O \cdot (1-n_1) = \pi_N \cdot n_1 + P_O \cdot (1-n_1) \quad \Rightarrow \quad n_1 = \frac{\pi_O - P_O}{\pi_N + \pi_O - P_N - P_O} \qquad (1)\ ;$$

$$P_N \cdot \nu_1 + \pi_N \cdot (1-\nu_1) = \pi_O \cdot \nu_1 + P_O \cdot (1-\nu_1) \quad \Rightarrow \quad \nu_1 = \frac{\pi_N - P_O}{\pi_N + \pi_O - P_N - P_O} \qquad (2)\ .$$

Alternatively, if "$\pi_N = \pi_O$" (game 2), then the complete-information Nash equilibrium

[1] These results were first presented by Chiappori, Levitt and Groseclose (2002).

solution of the game occurs when:

$$P_N \cdot n_2 + \pi_O \cdot (1-n_2) = \pi_O \cdot n_2 + P_O \cdot (1-n_2) \qquad \Rightarrow \qquad n_2 = \frac{\pi_O - P_O}{2 \cdot \pi_O - P_N - P_O} \qquad (3)\ ;$$

$$P_N \cdot \nu_2 + \pi_O \cdot (1-\nu_2) = \pi_O \cdot \nu_2 + P_O \cdot (1-\nu_2) \qquad \Rightarrow \qquad \nu_2 = \frac{\pi_O - P_O}{2 \cdot \pi_O - P_N - P_O} \qquad (4)\ .$$

The equilibrium values of n_1, n_2, v_1 and v_2 can be compared to obtain some relationships that will later be useful to analyze the Bayesian equilibrium of the corresponding incomplete-information game. By simple observation we find that "$n_2 = v_2$" (as we have already anticipated) and that "$n_2 > n_1$" (since both equilibrium expressions have the same numerator but n_1's denominator is greater than n_2's). We can also prove that "$v_1 > v_2$", as the following lemma shows.

Lemma 1: *Under complete information, the Nash equilibrium solution of the penalty-kick game implies that "$v_1 > v_2$".*

Proof: Suppose instead that "$v_1 < v_2$". Then it should hold that:

$$\nu_1 = \frac{\pi_N - P_O}{\pi_N + \pi_O - P_N - P_O} < \frac{\pi_O - P_O}{2 \cdot \pi_O - P_N - P_O} = \nu_2 \qquad .$$

But if this is so, then it should also hold that:

$$(\pi_N - P_O) \cdot (2 \cdot \pi_O - P_N - P_O) < (\pi_O - P_O) \cdot (\pi_N + \pi_O - P_N - P_O)\ ;$$

which implies that:

$$(2\pi_O - P_N)\pi_N - (2\pi_O + \pi_N)P_O + P_O(P_N + P_O) < (\pi_N + \pi_O - P_N)\pi_O - (2\pi_O + \pi_N)P_O + P_O(P_N + P_O)\ ;$$

$$(2 \cdot \pi_O - P_N) \cdot \pi_N < (\pi_N + \pi_O - P_N) \cdot \pi_O \qquad \Rightarrow \qquad (\pi_O - P_N) \cdot \pi_N < (\pi_O - P_N) \cdot \pi_O$$

$$\Rightarrow \qquad (\pi_O - P_N) \cdot (\pi_N - \pi_O) < 0\,.$$

But, as we know that "$\pi_O > P_N$" and "$\pi_N > \pi_O$", then this is a contradiction. Therefore it holds that "$v_1 > v_2$", q.e.d.

2.2. Incomplete information

Let us now turn to the incomplete-information case, in which the goalkeeper does not know if he is facing kicker 1 or kicker 2, but the kicker knows his type (and also the unique goalkeeper's type). In this case the goalkeeper will choose NS with some probability v_M, regardless of the fact that he is facing kicker 1 or kicker 2. Given this, kicker 1 is strictly better off by shooting NS if it holds that "$v_M < v_1$", while he is strictly better off by shooting OS if it holds that "$v_M > v_1$". Correspondingly, kicker 2 is strictly better off by shooting NS if it holds that "$v_M < v_2$", while he is strictly better off by shooting OS if it holds that "$v_M > v_2$".

Let us first assume that, as "$v_1 > v_2$", the goalkeeper chooses a value for v_M such that "$v_1 > v_M > v_2$". In this case kicker 1's best response will be to play NS as a pure strategy, and kicker 2's best response will be to play OS. But this could only be an equilibrium if the goalkeeper is indifferent between choosing NS and OS himself, for which it should hold that:

$$P_N \cdot \theta + \pi_O \cdot (1-\theta) = \pi_N \cdot \theta + P_O \cdot (1-\theta) \qquad \Rightarrow \qquad \theta = \frac{\pi_O - P_O}{\pi_N + \pi_O - P_N - P_O} \qquad (5)\ ;$$

and this is something that will generically occur with zero probability[2]. We should therefore look for alternative equilibria in which one of the kicker's types plays a pure strategy and the other one plays a mixed strategy. Two of those equilibria exist, and we will label them "case A" and "case B".

In case A, kicker 1 chooses NS, and both kicker 2 and the goalkeeper play mixed strategies. For this to occur, v_M has to be equal to v_2, and therefore kicker 1 is strictly better off by playing NS and kicker 2 is indifferent between NS and OS. For the goalkeeper to be indifferent between NS and OS, however, we need that:

$$P_N \cdot \theta + [P_N \cdot n_2 + \pi_O \cdot (1-n_2)] \cdot (1-\theta) = \pi_N \cdot \theta + [\pi_O \cdot n_2 + P_O \cdot (1-n_2)] \cdot (1-\theta)$$

$$\Rightarrow \qquad n_2 = \frac{(\pi_O - P_O) - (\pi_N - P_N) \cdot \theta / (1-\theta)}{2 \cdot \pi_O - P_N - P_O} \qquad (6)\ .$$

In case B, conversely, kicker 2 always chooses OS, and both kicker 1 and the goalkeeper play mixed strategies. For this to occur, v_M has to be equal to v_1, and therefore kicker 2 is strictly better off by playing OS and kicker 1 is indifferent between NS and OS. For the goalkeeper to be indifferent between NS and OS, however, we need that:

$$\pi_O \cdot (1-\theta) + [P_N \cdot n_1 + \pi_O \cdot (1-n_1)] \cdot \theta = P_O \cdot (1-\theta) + [\pi_N \cdot n_1 + P_O \cdot (1-n_1)] \cdot \theta$$

$$\Rightarrow \qquad n_1 = \frac{(\pi_O - P_O) / \theta}{\pi_N + \pi_O - P_N - P_O} \qquad (7)\ .$$

Both equilibria under cases A and B can also be seen as situations in which the goalkeeper is randomizing between NS and OS because he has the belief that one of the players is choosing a pure strategy with probability one, and the other player is playing a mixed strategy such as the one described by equations 6 or 7. Under case A, therefore, his belief is that, on average, the kicker will choose NS with a certain probability n_A equal to:

[2] This is because, as θ could be any real number between zero and one, then the probability that it is exactly equal to a particular real number is always zero (as there are infinite real numbers between zero and one).

$$n_A = \theta \cdot n_1 + (1-\theta)\cdot n_2 = \theta + (1-\theta)\frac{(\pi_O - P_O) - (\pi_N - P_N)\theta/(1-\theta)}{2\cdot \pi_O - P_N - P_O} = \frac{(\pi_O - P_O) - (\pi_N - P_N)\cdot\theta}{2\cdot \pi_O - P_N - P_O} \quad (8)\ ;$$

while under case B his belief is that, on average, the kicker will choose NS with a certain probability n_B equal to:

$$n_B = \theta\cdot n_1 + (1-\theta)\cdot n_2 = \theta\cdot\frac{(\pi_O - P_O)/\theta}{\pi_N + \pi_O - P_N - P_O} + (1-\theta)\cdot 0 = \frac{\pi_O - P_O}{\pi_N + \pi_O - P_N - P_O} \qquad (9)\ .$$

One interesting property of the Bayesian equilibria of this game under incomplete information is that, for a given set of parameters, only one of them exists. Indeed, the situation is such that, if "$\theta < (\pi_O$-$P_O)/(\pi_N$+π_O-P_N-$P_O)$", then case A equilibrium exists and case B equilibrium does not, while if "$\theta > (\pi_O$-$P_O)/(\pi_N$+π_O-P_N-$P_O)$", then case B equilibrium exists and case A equilibrium does not. These relationships are the results of the following propositions:

Proposition 1: *If the Bayesian equilibrium of the case A incomplete-information game exists, then it should hold that "$\theta < (\pi_O$-$P_O)/(\pi_N$+π_O-P_N-$P_O)$".*

Proof: Under the Bayesian equilibrium of case A, kicker 2 should play NS with a positive probability. Therefore it should hold that:

$$n_2 = \frac{(\pi_O - P_O) - (\pi_N - P_N)\cdot\theta/(1-\theta)}{2\cdot\pi_O - P_N - P_O} > 0 \qquad .$$

But if this is so, then it should also hold that:

$$\pi_O - P_O > (\pi_N - P_N)\cdot\theta/(1-\theta) \qquad \Rightarrow \qquad (\pi_O - P_O)\cdot(1-\theta) > (\pi_N - P_N)\cdot\theta \qquad ;$$

which implies that:

$$\pi_O - P_O > (\pi_N + \pi_O - P_N - P_O)\cdot\theta \qquad \Rightarrow \qquad \theta < \frac{\pi_O - P_O}{\pi_N + \pi_O - P_N - P_O} \qquad \text{q.e.d.}$$

Proposition 2: *If the Bayesian equilibrium of the case B incomplete-information game exists, then it should hold that "$\theta > (\pi_O$-$P_O)/(\pi_N$+π_O-P_N-$P_O)$".*

Proof: Under the Bayesian equilibrium of case B, kicker 1 should play NS with a probability that is smaller than one. Therefore it should hold that:

$$n_1 = \frac{(\pi_O - P_O)/\theta}{\pi_N + \pi_O - P_N - P_O} < 1 \qquad .$$

But if this is so, then it should also hold that:

$$\pi_O - P_O < (\pi_N + \pi_O - P_N - P_O)\cdot\theta \qquad \Rightarrow \qquad \theta > \frac{\pi_O - P_O}{\pi_N + \pi_O - P_N - P_O} \qquad \text{q.e.d.}$$

The model described in the previous paragraphs implies that, when there are two types of kickers and only one type of goalkeeper, and there is incomplete information, then the Bayesian equilibrium of the corresponding incomplete-information game generally implies that one of the kicker's types will choose a pure strategy and the other type will choose a mixed strategy, while the goalkeeper will also choose a mixed strategy (which is the same strategy that he would choose if he were facing the kicker who is playing a mixed strategy). If we compare this Bayesian equilibrium with the Nash equilibria that would occurred if the same games were played under complete information, we would see that in this situation the goalkeeper is typically worse off and one of the kicker's types is typically better off.

In order to perform the comparisons outlined in the previous paragraph, we should compare the expected scoring probabilities under different situations. From those comparisons we will see that, given the parameters that we use in our model, kicker 1 obtains a higher expected payoff (i.e., a higher expected scoring probability) than kicker 2 under a complete-information Nash equilibrium. When we turn to the incomplete-information Bayesian equilibria analyzed, we see that the kicker who chooses a mixed strategy obtains the same expected payoff than under complete information, while the kicker who chooses a pure strategy is strictly better off. Under case B, moreover, kicker 2 is able to obtain the same expected payoff than kicker 1.

The expected scoring probability of a particular kicker is simply the average of the scoring probabilities implied by the strategy that he chooses, weighted by the probabilities that the goalkeeper "guesses" that strategy and by the probability that the goalkeeper "does not guess" that strategy. When a kicker is playing a mixed strategy, then the expected scoring probability of both NS and OS should be the same. When he is playing a pure strategy, conversely, his expected scoring probability is the one associated to the pure strategy that he chooses, that has to be greater than the expected scoring probability of the alternative strategy.

Under complete information, kicker 1's expected scoring probability is equal to:

$$SP_1(CI) = P_N \cdot \nu_1 + \pi_N \cdot (1-\nu_1) = \pi_O \cdot \nu_1 + P_O \cdot (1-\nu_1) = \frac{\pi_N \cdot \pi_O - P_N \cdot P_O}{\pi_N + \pi_O - P_N - P_O} \quad (10)\ ;$$

while kicker 2's expected scoring probability is equal to:

$$SP_2(CI) = P_N \cdot \nu_2 + \pi_O \cdot (1-\nu_2) = \pi_O \cdot \nu_2 + P_O \cdot (1-\nu_2) = \frac{{\pi_O}^2 - P_N \cdot P_O}{2 \cdot \pi_O - P_N - P_O} \quad (11)\ .$$

Under incomplete information, the expected scoring probabilities for the kickers depend on the case that holds. Under case A Bayesian equilibrium, kicker 2 obtains the same expected scoring probability that he gets under complete information, because "$\nu_A = \nu_2$" and therefore he is indifferent between choosing NS and OS. Kicker 1, conversely, is strictly better off by choosing NS, which now gives him the following expected scoring probability:

$$SP_1(IA) = P_N \cdot \nu_A + \pi_A \cdot (1-\nu_A) = \frac{\pi_O \cdot (\pi_N + P_N) - P_N \cdot (\pi_N + P_O)}{2 \cdot \pi_O - P_N - P_O} \qquad (12)\ .$$

Conversely, under case B, kicker 1 obtains the same expected scoring probability that he gets under complete information, because "$\nu_B = \nu_1$" and therefore he is indifferent between choosing NS and OS. The one who is strictly better off is kicker 2, who is now choosing OS and obtaining the following expected scoring probability:

$$SP_2(IB) = \pi_O \cdot \nu_B + P_O \cdot (1-\nu_B) = \frac{\pi_O \cdot (\pi_N + P_O) - P_O \cdot (\pi_O + P_N)}{\pi_N + \pi_O - P_N - P_O} = \frac{\pi_N \cdot \pi_O - P_N \cdot P_O}{\pi_N + \pi_O - P_N - P_O} \qquad (13)\ .$$

The idea that kicker 1 is better off than kicker 2 under complete information comes from the fact that, under the assumptions used in this paper, both kickers obtain the same expected payoff in three of the four cells of the scoring-probability matrix (see table 1) while kicker 1 gets a higher payoff in the remaining cell (since "$\pi_N > \pi_O$"). As the goalkeeper adjusts his strategy to this situation, however, the relationship between the expected scoring probabilities that these two types of kickers induce is not so obvious when one observes the equilibrium values gotten at equations 10 and 11. The proof that $SP_1(CI)$ is actually greater than $SP_2(CI)$, therefore, is given in the following proposition.

Proposition 3: *Under complete information, the expected scoring probability for kicker 1 is greater than the expected scoring probability for kicker 2.*

Proof: Under complete information, the expected scoring probability for kicker 1 is the same choosing NS and OS. Similarly, the expected scoring probability for kicker 2 is the same choosing NS and OS. Therefore we can write that:

$$SP_1(CI) = SP_1(CI/OS) = \pi_O \cdot \nu_1 + P_O \cdot (1-\nu_1) = P_O + (\pi_O - P_O) \cdot \nu_1 \quad ;$$

$$SP_2(CI) = SP_2(CI/OS) = \pi_O \cdot \nu_2 + P_O \cdot (1-\nu_2) = P_O + (\pi_O - P_O) \cdot \nu_2 \quad .$$

As we assume that "$\pi_O > P_O$", and we know from lemma 1 that "$\nu_1 > \nu_2$", then we also know that:

$$P_O + (\pi_O - P_O) \cdot \nu_1 > P_O + (\pi_O - P_O) \cdot \nu_2 \qquad \Rightarrow \qquad SP_1(CI) > SP_2(CI) \qquad \text{q.e.d.}$$

A second comparison that we can make between expected scoring probabilities is the one that refers to $SP_1(CI)$ and $SP_1(IA)$, which is the theme of proposition 4. Finally, we can also prove that "$SP_2(IB) > SP_2(CI)$", and this is the theme of proposition 5.

Proposition 4: *Under case A Bayesian equilibrium with incomplete information, the expected scoring probability for kicker 1 is greater than the one that he obtains under complete information.*

Proof: Under complete information, the expected scoring probability for kicker 1 is the same choosing NS and OS. Conversely, the expected scoring probability for kicker 1 under case A with incomplete information is greater if he chooses NS, which is the pure strategy that he actually chooses in equilibrium. Therefore we can write that:

$$SP_1(CI) = SP_1(CI/NS) = P_N \cdot \nu_1 + \pi_N \cdot (1-\nu_1) = \pi_N - (\pi_N - P_N) \cdot \nu_1 \quad ;$$

$$SP_1(IA) = SP_1(IA/NS) = P_N \cdot \nu_A + \pi_N \cdot (1-\nu_A) = \pi_N - (\pi_N - P_N) \cdot \nu_A \quad .$$

By the definition of case A Bayesian equilibrium, we know that "$\nu_A = \nu_2$". As we also know that "$\pi_N > P_N$" (by assumption) and "$\nu_1 > \nu_2$" (from lemma 1), then is should hold that:

$$\pi_N - (\pi_N - P_N) \cdot \nu_1 < \pi_N - (\pi_N - P_N) \cdot \nu_A \quad \Rightarrow \quad SP_1(CI) < SP_1(IA) \quad \text{q.e.d.}$$

Proposition 5: *Under case B Bayesian equilibrium with incomplete information, the expected scoring probability for kicker 2 is greater than the one that he obtains under complete information.*

Proof: Under case B with incomplete information, the expected scoring probability for kicker 2 ($SP_2(IB)$) is the same than the expected scoring probability for kicker 1 under complete information ($SP_1(CI)$), since they are both equal to "$(\pi_N \cdot \pi_O - P_N \cdot P_O)/(\pi_N + \pi_O - P_N - P_O)$". As we know (from proposition 3) that "$SP_1(CI) > SP_2(CI)$", then this implies that "$SP_2(IB) > SP_2(CI)$", q.e.d.

3. A model with three strategies

3.1. Complete information

Let us now assume that our two players (i.e., the goalkeeper and the kicker) can choose among three different strategies instead of two. The third additional strategy (besides the kicker's natural side and the kicker's opposite side) is the center of the goal (C), so the kicker can now choose to shoot to that place and the goalkeeper can choose to stay in that place. Following Coloma (2007), we will use the letter μ to denote the probability of scoring if the kicker chooses to shoot C and the goalkeeper chooses either the kicker's natural side (NS) or the kicker's opposite side (OS), and we will also assume that, if both the goalkeeper and the kicker choose C, then the scoring probability is zero[3].

Using the same assumptions derived from the theoretical and empirical literature about this topic, we will now assume that "$\pi_N \geq \pi_O > \mu > P_N > P_O > 0$", which implies that the scoring probability when the kicker shoots to the center of the goal (and the goalkeeper

[3] These assumptions, in fact, are inherited from the model originally proposed by Chiappori, Levitt and Groseclose (2002).

chooses either NS or OS) is lower than the scoring probability of choosing any of the sides (when the goalkeeper chooses the other side or the center) but higher than the scoring probability of a situation in which the ball goes to one of the sides and the goalkeeper guesses the shot. We will also assume that there are two types of kicker (kicker 1 and kicker 2) and a single type of goalkeeper. Those kickers are characterized by having different values for the parameter μ, such that "$\mu_1 < \mu_2$". The corresponding probability matrix, therefore, is the one that appears on table 2.

Table 2: Scoring-probability matrix for the three-strategy game

		Goalkeeper		
		NS	C	OS
Kicker 1 (Prob θ)	NS	P_N	π_N	π_N
	C	μ_1	0	μ_1
	OS	π_O	π_O	P_O
Kicker 2 (Prob 1-θ)	NS	P_N	π_N	π_N
	C	μ_2	0	μ_2
	OS	π_O	π_O	P_O

Let us assume, moreover, that the values of μ_1 and μ_2 are such that:

$$\mu_1 < \frac{\pi_N \cdot \pi_O - P_N \cdot P_O}{\pi_N + \pi_O - P_N - P_O} \quad ; \quad \mu_2 > \frac{\pi_N \cdot \pi_O - P_N \cdot P_O}{\pi_N + \pi_O - P_N - P_O} \qquad (14)\ .$$

Following Chiappori, Levitt and Groseclose (2002), we know that in that case the corresponding complete-information Nash equilibria occur when it holds that (game 1):

$$n_1 = \frac{\pi_O - P_O}{\pi_N + \pi_O - P_N - P_O} \quad ; \quad c_1 = 0 \qquad (15)\ ;$$

$$\nu_1 = \frac{\pi_N - P_O}{\pi_N + \pi_O - P_N - P_O} \quad ; \quad \gamma_1 = 0 \qquad (16)\ ;$$

and when it holds that (game 2):

$$n_2 = \frac{\mu_2 \cdot (\pi_O - P_O)}{(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \quad ;$$

$$c_2 = \frac{(\pi_N - P_N) \cdot (\pi_O - P_O)}{(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \qquad (17)\ ;$$

$$\nu_2 = \frac{\pi_N \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N - \pi_O)}{(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \quad ;$$

$$\gamma_2 = \frac{\mu_2 \cdot (\pi_N + \pi_O - P_N - P_O) + P_N \cdot P_O - \pi_N \cdot \pi_O}{(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \qquad (18) ;$$

where c_1 is the probability that kicker 1 chooses C, c_2 is the probability that kicker 2 chooses C, γ_1 is the probability that the goalkeeper chooses C when he faces kicker 1, and γ_2 is the probability that the goalkeeper chooses C when he faces kicker 2.

As we see, the fact that μ_1 is a relatively small number induces kicker 1 not to choose C in any circumstance (i.e., it makes C a dominated strategy for kicker 1). Knowing that, the goalkeeper never chooses C, either, when facing kicker 1. Conversely, as μ_2 is relatively large, kicker 2 is willing to choose C with some positive probability. Knowing that, the goalkeeper sometimes chooses C when facing kicker 2. Following the terminology of Chiappori, Levitt and Groseclose (2002), we will say that the Nash equilibrium of the game between kicker 1 and the goalkeeper is a "restricted randomization equilibrium", while the Nash equilibrium of the game between kicker 2 and the goalkeeper is a "general randomization equilibrium".

Due to the fact that in this model we are assuming that "$\mu_1 < \mu_2$" (and all the other parameters are the same for the two types of kicker), then we will also conclude that the expected scoring probability under game 1 will be smaller than the expected scoring probability under game 2. This is in fact the result of the following lemma.

Lemma 2: *Under complete information, the expected scoring probability for kicker 1 is smaller than the expected scoring probability for kicker 2.*

Proof: Substituting the equilibrium values of v_1, v_2 and γ_2 into the expected scoring probabilities of kickers 1 and 2 when they either choose NS, OS or C, we can write that:

$$SP_1(NS) = SP_1(OS) = \frac{\pi_N \cdot \pi_O - P_N \cdot P_O}{\pi_N + \pi_O - P_N - P_O} \qquad ;$$

$$SP_2(NS) = SP_2(OS) = SP_2(C) = \frac{\mu_2 \cdot [\pi_N \cdot (\pi_O - P_O) + \pi_O \cdot (\pi_N - P_N)]}{(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \qquad .$$

If we assumed that "$SP_1 > SP_2$", then it should hold that:

$$\frac{\pi_N \cdot \pi_O - P_O \cdot P_N}{\pi_N + \pi_O - P_N - P_O} > \frac{\mu_2 \cdot [\pi_N \cdot (\pi_O - P_O) + \pi_O \cdot (\pi_N - P_N)]}{(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \qquad \Rightarrow$$

$$(\pi_N \cdot \pi_O - P_O \cdot P_N) \cdot (\pi_N - P_N) \cdot (\pi_O - P_O) > \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O) \cdot (\pi_N - P_N) \cdot (\pi_O - P_O) \qquad \Rightarrow$$

$$\mu_2 < \frac{\pi_N \cdot \pi_O - P_N \cdot P_O}{\pi_N + \pi_O - P_N - P_O} \qquad ;$$

but this is a contradiction with the assumption stated in equation 14. Therefore it holds that "$SP_1 < SP_2$", q.e.d.

3.2. Incomplete information

Let us now turn to an incomplete-information case, in which the goalkeeper does not know if he is facing kicker 1 or kicker 2, but the kicker knows his type (and also the unique goalkeeper's type). Let us assume that there is a probability θ that the goalkeeper faces kicker 1, and a probability $1-\theta$ that he faces kicker 2. In that case, as we did for the two-strategy model, we have to look for a Bayesian equilibrium in which the goalkeeper chooses a single strategy and each of the possible kickers chooses his own strategy.

One possible Bayesian equilibrium for this situation (case A) occurs when the goalkeeper chooses the same strategy that he would choose in a complete-information setting in which he were facing kicker 1 (i.e., $\nu_A > 0$, $\gamma_A = 0$). Given that, kicker 1 is indifferent between choosing NS and OS, and kicker 2 is strictly better-off by choosing C, provided that the goalkeeper never chooses C in his equilibrium strategy. All these results can be stated as follows:

$$n_1 = \frac{\pi_O - P_O}{\pi_N + \pi_O - P_N - P_O} \quad ; \quad c_1 = 0 \qquad (19) ;$$

$$n_2 = 0 \quad ; \quad c_2 = 1 \qquad (20) ;$$

$$\nu_A = \frac{\pi_N - P_O}{\pi_N + \pi_O - P_N - P_O} \quad ; \quad \gamma_A = 0 \qquad (21) .$$

Another possible Bayesian equilibrium (case B) occurs when the goalkeeper chooses the same strategy that he would choose in a complete-information setting in which he were facing kicker 2 (i.e., $\nu_B > 0$, $\gamma_B > 0$). Given that, kicker 2 is indifferent between choosing NS, OS or C, and kicker 1 is indifferent between choosing NS or OS[4]. This implies that:

$$\nu_B = \frac{\pi_N \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N - \pi_O)}{(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \quad ;$$

$$\gamma_B = \frac{\mu_2 \cdot (\pi_N + \pi_O - P_N - P_O) + P_N \cdot P_O - \pi_N \cdot \pi_O}{(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \qquad (22) ;$$

$$SP_1(NS) = SP_1(OS) = \frac{\mu_2 \cdot [\pi_N \cdot (\pi_O - P_O) + \pi_O \cdot (\pi_N - P_N)]}{(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \qquad (23) ;$$

[4] This last feature has to do with the fact that, in our model, both kickers have the same values for "π_N", "π_O", "P_N" and "P_O". If there were some differences in these values for the two types of kickers, then kicker 1 might strictly prefer either NS or OS.

$$SP_2(NS) = SP_2(OS) = SP_2(C) = \frac{\mu_2 \cdot [\pi_N \cdot (\pi_O - P_O) + \pi_O \cdot (\pi_O - P_N)]}{(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \quad (24) .$$

The values for n_1 and n_2 in this Bayesian equilibrium, however, are indeterminate, since what we need is that, on average, they equate the value that n_2 has in the corresponding complete-information Nash equilibrium. The equilibrium value for c_2, conversely, is a function of the probability parameter θ. Indeed:

$$\theta \cdot n_1 + (1-\theta) \cdot n_2 = \frac{\mu_2 \cdot (\pi_O - P_O)}{(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \quad (25) ;$$

$$(1-\theta) \cdot c_2 = \frac{(\pi_N - P_N) \cdot (\pi_O - P_O)}{(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \quad \Rightarrow$$

$$c_2 = \frac{(\pi_N - P_N) \cdot (\pi_O - P_O)}{(1-\theta) \cdot [(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)]} \quad (26) .$$

For these strategy profiles to be Bayesian equilibria, however, some additional conditions have to be fulfilled. Under case A, for example, we need that the goalkeeper be indifferent between choosing NS and OS, and strictly better-off by choosing any of those strategies than by choosing C. Let us now define the corresponding expected scoring probabilities induced by the three possible goalkeeper strategies (NS, OS and C) in the following way:

$$SP_G(NS) = \theta \cdot [n_1 \cdot P_N + (1-n_1) \cdot \pi_O] + (1-\theta) \cdot \mu_2 = \frac{\theta \cdot (\pi_N \cdot \pi_O - P_N \cdot P_O)}{\pi_N + \pi_O - P_N - P_O} + (1-\theta) \cdot \mu_2 \quad (27) ;$$

$$SP_G(OS) = \theta \cdot [n_1 \cdot \pi_N + (1-n_1) \cdot P_O] + (1-\theta) \cdot \mu_2 = \frac{\theta \cdot (\pi_N \cdot \pi_O - P_N \cdot P_O)}{\pi_N + \pi_O - P_N - P_O} + (1-\theta) \cdot \mu_2 \quad (28) ;$$

$$SP_G(C) = \theta \cdot [n_1 \cdot \pi_N + (1-n_1) \cdot \pi_O] + (1-\theta) \cdot 0 = \frac{\theta \cdot (2 \cdot \pi_N \cdot \pi_O - \pi_N \cdot P_O - \pi_O \cdot P_N)}{\pi_N + \pi_O - P_N - P_O} \quad (29) .$$

As we see, the equilibrium values found for n_1 and c_2 imply that in this case $SP_G(NS)$ and $SP_G(OS)$ are equal, so the goalkeeper is actually indifferent between choosing NS and OS. We will also need that, in this equilibrium, $SP_G(NS)$ and $SP_G(OS)$ are greater than $SP_G(C)$, but this only occurs for a set of values of the parameter θ, as the following proposition shows.

Proposition 6: *If the Bayesian equilibrium of the case A incomplete-information game exists, then it should hold that* "$\theta > \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O) / [(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)]$".

Proof: Under the Bayesian equilibrium of case A, the goalkeeper should strictly prefer to play both NS and OS with a positive probability instead of C. Therefore it should hold that:

$$SP_G(C) > SP_G(NS) = SP_G(OS) \qquad \Rightarrow$$

$$\frac{\theta\cdot(2\cdot\pi_N\cdot\pi_O - \pi_N\cdot P_O - \pi_O\cdot P_N)}{\pi_N + \pi_O - P_N - P_O} > \frac{\theta\cdot(\pi_N\cdot\pi_O - P_N\cdot P_O)}{\pi_N + \pi_O - P_N - P_O} + (1-\theta)\cdot\mu_2 \qquad \Rightarrow$$

$$\theta > \frac{\mu_2\cdot(\pi_N + \pi_O - P_N - P_O)}{(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2\cdot(\pi_N + \pi_O - P_N - P_O)} \qquad \text{q.e.d.}$$

On the other hand, for a case B equilibrium to exist, it is important that n_1, n_2 and c_2 take some values that are not inconsistent with their status as probability values. In particular, we need that c_2 is not greater than one, and this also occurs for a particular set of values of the parameter θ. This is the theme of proposition 7.

Proposition 7: *If the Bayesian equilibrium of the case B incomplete-information game exists, then it should hold that* "$\theta < \mu_2\cdot(\pi_N+\pi_O-P_N-P_O)/[(\pi_N-P_N)\cdot(\pi_O-P_O)+\mu_2\cdot(\pi_N+\pi_O-P_N-P_O)]$".

Proof: Under the Bayesian equilibrium of case B, kicker 2 should choose C with a certain probability (c_2) that guarantees that the goalkeeper is indifferent between choosing NS, OS and C. But this can only be feasible if the required equilibrium value for c_2 is less than one. Therefore it should hold that:

$$c_2 = \frac{(\pi_N - P_N)\cdot(\pi_O - P_O)}{(1-\theta)\cdot[(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2\cdot(\pi_N + \pi_O - P_N - P_O)]} < 1 \qquad \Rightarrow$$

$$1-\theta > \frac{(\pi_N - P_N)\cdot(\pi_O - P_O)}{(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2\cdot(\pi_N + \pi_O - P_N - P_O)} \qquad \Rightarrow$$

$$\theta < \frac{\mu_2\cdot(\pi_N + \pi_O - P_N - P_O)}{(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2\cdot(\pi_N + \pi_O - P_N - P_O)} \qquad \text{q.e.d.}$$

Note that propositions 6 and 7 imply that case A and case B equilibria cannot exist at the same time. Indeed, for any particular value of θ, only one of these equilibria can occur, being case A equilibrium the chosen one when θ is relatively large, and case B equilibrium the chosen one when θ is relatively small.

Another set of restrictions on parameter θ can be found if we analyze the possible values of n_1 and n_2 under a case B equilibrium. Recall that, from equation 26, we know that n_1 and n_2 have to be such that, on average, they have a value equal to $\mu_2\cdot(\pi_O- P_O)/[(\pi_N-P_N)\cdot(\pi_O-P_O)+\mu_2\cdot(\pi_N+\pi_O-P_N-P_O)]$. But the possible combinations of n_1 and n_2 that satisfy that equation are also limited by the conditions that "$0 \le n_1 \le 1$", and "$0 \le n_2 \le 1-c_2$". In the particular cases where one of these constraints holds as an equality, then the other strategy coefficient adopts a determinate value. But this value is also constrained by some restrictions, and this imposes a limit on the possible value for the parameter θ. In particular:

$$n_1 = 0 \quad \Rightarrow \quad n_2 = \frac{\mu_2 \cdot (\pi_O - P_O)}{(1-\theta)\cdot[(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)]} < 1 - c_2 \;\Rightarrow$$

$$\frac{\mu_2(\pi_O - P_O)}{(1-\theta)[(\pi_N - P_N)(\pi_O - P_O) + \mu_2(\pi_N + \pi_O - P_N - P_O)]} < \frac{(1-\theta)\mu_2(\pi_N + \pi_O - P_N - P_O) - \theta(\pi_N - P_N)(\pi_O - P_O)}{(1-\theta)[(\pi_N - P_N)(\pi_O - P_O) + \mu_2(\pi_N + \pi_O - P_N - P_O)]}$$

$$\Rightarrow \quad \theta\cdot[(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)] < \mu_2 \cdot (\pi_N - P_N)$$

$$\Rightarrow \quad \theta < \frac{\mu_2 \cdot (\pi_N - P_N)}{(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \qquad (30)\ ;$$

$$n_1 = 1 \quad \Rightarrow \quad n_2 = \frac{\mu_2 \cdot (\pi_O - P_O) - \theta\cdot[(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)]}{(1-\theta)\cdot[(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)]} > 0$$

$$\Rightarrow \quad \theta < \frac{\mu_2 \cdot (\pi_O - P_O)}{(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \qquad (31)\ ;$$

$$n_2 = 0 \quad \Rightarrow \quad n_1 = \frac{\mu_2 \cdot (\pi_O - P_O)}{\theta\cdot[(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)]} < 1$$

$$\Rightarrow \quad \theta > \frac{\mu_2 \cdot (\pi_O - P_O)}{(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \qquad (32)\ ;$$

$$n_2 = 1 - c_2 \quad \Rightarrow \quad n_1 = \frac{\theta\cdot[(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)] - \mu_2 \cdot (\pi_N - P_N)}{\theta\cdot[(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)]} > 0$$

$$\Rightarrow \quad \theta > \frac{\mu_2 \cdot (\pi_N - P_N)}{(\pi_N - P_N)\cdot(\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \qquad (33)\ .$$

Although in our case B model all these restrictions apply to particular situations where either n_1 or n_2 adopt extreme values, those situations are actually the only ones that are feasible if the parameters π_N , π_O , P_N and P_O are not exactly the same for kickers 1 and 2. Imagine, for example, that one of these parameters is slightly larger or smaller for kicker 1, and that this difference induces that kicker to strictly prefer NS when the goalkeeper responds optimally to kicker 2. In that case, "$n_1 = 1$" (if it holds that "$\theta < \mu_2\cdot(\pi_O - P_O)/[(\pi_N-P_N)\cdot(\pi_O-P_O)+\mu_2\cdot(\pi_N+\pi_O-P_N-P_O)]$"), or else "$n_2 = 0$" (if it holds that "$\theta > \mu_2\cdot(\pi_O - P_O)/[(\pi_N-P_N)\cdot(\pi_O-P_O)+\mu_2\cdot(\pi_N+\pi_O-P_N-P_O)]$"). Conversely, if the parameter values are such that kicker 1 strictly prefers OS when the goalkeeper responds optimally to kicker 2, then the only possible case B equilibria occur when "$n_1 = 0$" (if it holds that "$\theta < \mu_2 \cdot (\pi_N - P_N)/[(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)]$"), or else when "$n_2 = 1 - c_2$" (if it holds that "$\theta > \mu_2 \cdot (\pi_N - P_N)/[(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)]$")[5].

An additional group of results that we can find using this incomplete-information setting has to do with the idea that the expected scoring probabilities are higher under incomplete information than under complete information. This is particularly the case for kicker 1 under case B, since we have found (equation 23) that in this circumstance he obtains the same expected scoring probability than kicker 2 under complete information, and by lemma 2 we know that such probability is greater than the one that kicker 1 obtains under complete information. Another case in which a kicker's type obtains a strictly higher scoring probability with incomplete information is the one of kicker 2 under case A, as is proved in lemma 3.

Lemma 3: *Under case A Bayesian equilibrium with incomplete information, the expected scoring probability for kicker 2 is greater than the one that he obtains under complete information.*

Proof: Suppose instead that "$SP_2(CI) > SP_2(IA)$". Then it should hold that:

$$SP_2(CI) = \frac{\mu_2 \cdot [\pi_N \cdot (\pi_O - P_O) + \pi_O \cdot (\pi_N - P_N)]}{(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} > \mu_2 = SP_2(IA) \quad .$$

But if this is so, then it should also hold that:

$$\mu_2 \cdot (\pi_N \cdot \pi_O - P_N \cdot P_O) > \mu_2{}^2 \cdot (\pi_N + \pi_O - P_N - P_O) \qquad \Rightarrow \qquad \mu_2 < \frac{\pi_N \cdot \pi_O - P_N \cdot P_O}{\pi_N + \pi_O - P_N - P_O} \quad .$$

As we know from equation 14 that this last result is not true, then this is a contradiction. Therefore, "$SP_2(CI) < SP_2(IA)$", q.e.d.

With all these results at hand, it is straightforward to prove that the average expected scoring probability is always higher under incomplete information, provided that "$0 < \theta < 1$". That is the theme of proposition 8.

Proposition 8: *If "$0 < \theta < 1$", then the average expected scoring probability is greater under incomplete information than under complete information*.

Proof: Recall that the expected scoring probabilities for the two types of kickers under the different analyzed cases are the following:

$$SP_1(CI) = SP_1(IA) = \frac{\pi_N \cdot \pi_O - P_N \cdot P_O}{\pi_N + \pi_O - P_N - P_O} \quad ; \qquad SP_2(IA) = \mu_2 \quad ;$$

$$SP_2(CI) = SP_2(IB) = SP_1(IB) = \frac{\mu_2 \cdot [\pi_N \cdot (\pi_O - P_O) + \pi_O \cdot (\pi_N - P_N)]}{(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \quad .$$

[5] In fact, these conditions are additional to the general requirement that "$\theta < \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)/[(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)]$", which is the one that guarantees that a case B Bayesian equilibrium exists.

Let us now define the average expected scoring probabilities in the following way:

$$\mathrm{ASP(CI)} = \theta \cdot \frac{\pi_N \cdot \pi_O - P_N \cdot P_O}{\pi_N + \pi_O - P_N - P_O} + (1-\theta) \cdot \frac{\mu_2 \cdot [\pi_N \cdot (\pi_O - P_O) + \pi_O \cdot (\pi_N - P_N)]}{(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \quad ;$$

$$\mathrm{ASP(IA)} = \theta \cdot \frac{\pi_N \cdot \pi_O - P_N \cdot P_O}{\pi_N + \pi_O - P_N - P_O} + (1-\theta) \cdot \mu_2 \quad ;$$

$$\mathrm{ASP(IB)} = \frac{\mu_2 \cdot [\pi_N \cdot (\pi_O - P_O) + \pi_O \cdot (\pi_N - P_N)]}{(\pi_N - P_N) \cdot (\pi_O - P_O) + \mu_2 \cdot (\pi_N + \pi_O - P_N - P_O)} \quad .$$

As we know (from lemma 3) that "$SP_2(IA) > SP_2(CI)$", then we also know that "$ASP(IA) > ASP(CI)$". And as we know (from lemma 2) that "$SP_2(CI) = SP_1(IB) > SP_1(CI)$", then we also know that "$ASP(IB) > ASP(CI)$". Combining both results, it holds that, for any value of θ such that "$0 < \theta < 1$", it is true that "$ASP(II) > ASP(CI)$", q.e.d.

4. Numerical example

4.1. Two-strategy model

The results that we have obtained in section 2 can be illustrated for a particular set of parameters. Using the estimates that appear in Coloma (2007), we will assume that "$\pi_N = 0.98$", "$\pi_O = 0.94$", "$P_N = 0.68$" and "$P_O = 0.48$". This implies that, under complete information, the equilibrium values for n_1, n_2, ν_1 and ν_2 are the following:

$$n_1 = \frac{0.94 - 0.48}{0.98 + 0.94 - 0.68 - 0.48} = 0.6053 \quad ; \qquad \nu_1 = \frac{0.98 - 0.48}{0.98 + 0.94 - 0.68 - 0.48} = 0.6579 \quad ;$$

$$n_2 = \nu_2 = \frac{0.94 - 0.48}{2 \cdot 0.94 - 0.68 - 0.48} = 0.6389 \quad ;$$

which is therefore an example of the theoretical result that we obtained, which states that "$\nu_1 > \nu_2 = n_2 > n_1$". Besides, the corresponding expected scoring probabilities under this complete-information situation are the following:

$$\mathrm{SP_1(CI)} = \frac{0.98 \cdot 0.94 - 0.68 \cdot 0.48}{0.98 + 0.94 - 0.68 - 0.48} = 0.7826 \quad ; \qquad \mathrm{SP_2(CI)} = \frac{0.94^2 - 0.68 \cdot 0.48}{2 \cdot 0.94 - 0.68 - 0.48} = 0.7739 \quad .$$

If we now turn to the incomplete-information situation, we have two possible cases depending on the fact that θ is either greater than or smaller than 0.6053. When "$\theta < 0.6053$" (case A), it will hold that:

$$n_1 = 1 \quad ; \qquad \nu_A = \nu_2 = 0.6389 \quad ; \qquad n_2 = 0.6389 - \frac{0.4167 \cdot \theta}{1-\theta} \quad ;$$

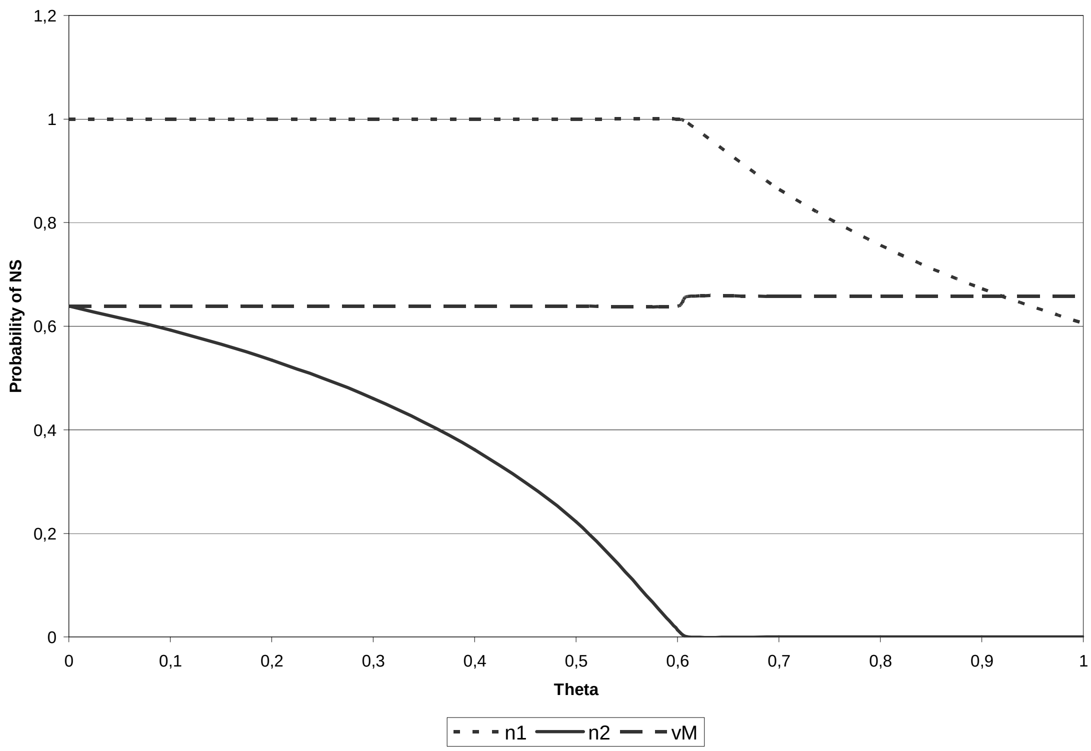
whereas, if "$\theta > 0.6053$" (case B), it will hold that:

$$n_2 = 0 \quad ; \qquad \nu_B = \nu_1 = 0.6579 \quad ; \qquad n_1 = \frac{0.6053}{\theta} \quad .$$

As we already know from the results obtained in section 2, "$SP_2(IA) = SP_2(CI) = 0.7739$" and "$SP_2(IB) = SP_1(IB) = SP_1(CI) = 0.7826$". By applying the formula that we have derived for equation 12, we can also find that:

$$SP_1(IA) = \frac{0.94 \cdot (0.98 + 0.68) - 0.68 \cdot (0.98 + 0.48)}{2 \cdot 0.94 - 0.68 - 0.48} = 0.7883 \quad .$$

Figure 1: Equilibrium strategies under incomplete information (two-strategy model)

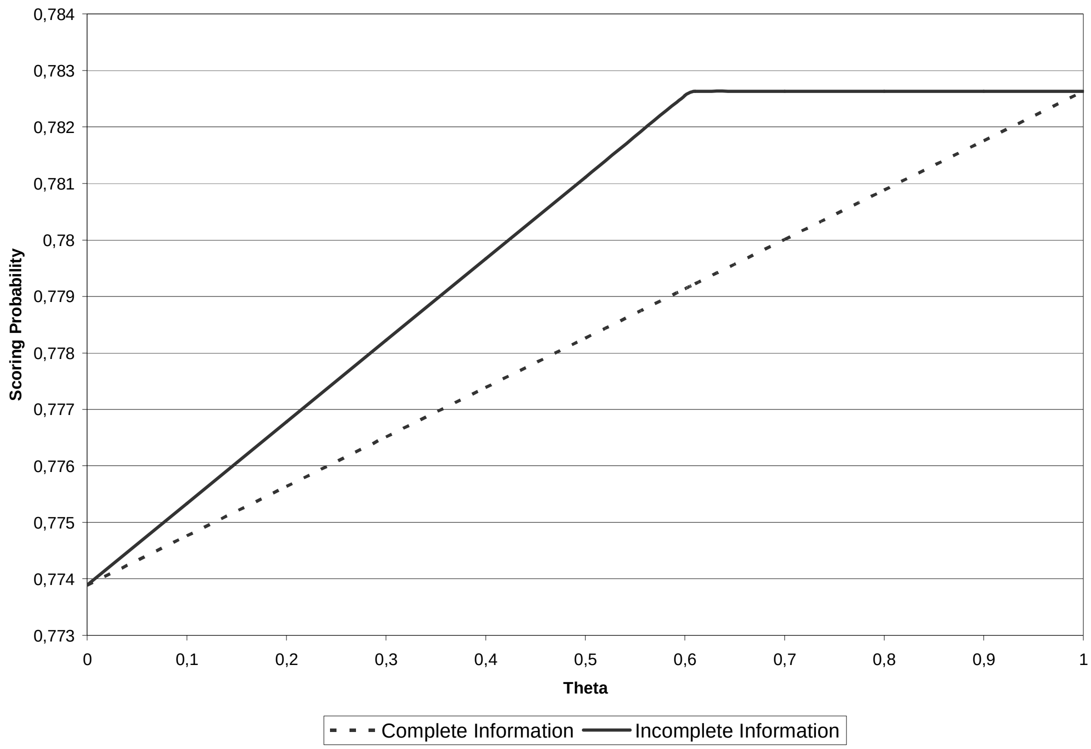


The incomplete-information case produces, as we have already seen, some results that depend on the value of θ, that is, on the proportion of kicker 1's that we have in the population under analysis. Figure 1 depicts the values of n_1, n_2 and v_M that we obtain as equilibrium values for all possible levels of θ, and in that figure we can see that n_1 tends to its complete-information level when θ tends to one, while n_2 tends to its complete-information level when θ tends to zero.

Correspondingly, figure 2 depicts the average scoring probability under complete and incomplete information for all possible levels of θ. In it we see that, unless "$\theta = 0$" or "$\theta = 1$", the average scoring probability is higher under incomplete information. We also see that, when

θ increases, the average scoring probability under complete information also increases (since "$SP_1(CI) > SP_2(CI)$", and the average scoring probability is "$\theta \cdot SP_1(CI) + (1-\theta) \cdot SP_2(CI)$"). The average scoring probability is also increasing in θ under incomplete information, but it reaches a maximum of 0.7826 when "$\theta = 0.6053$", and keeps that level for all values of θ that exceed that number.

Figure 2: Average scoring probabilities (two-strategy model)



4.2. Three-strategy model

If we now turn to the three-strategy model developed in section 3, we can also illustrate its results using the set of parameters that we applied in the previous sub-section of this paper. We will additionally need a value for μ_2, which can also be the one estimated in Coloma (2007). That value is "$\mu_2 = 0.88$", which, together with the values reported in sub-section 4.1, implies that under complete information:

$$n_1 = \frac{0.94-0.48}{0.98+0.94-0.68-0.48} = 0.6053 \quad ; \qquad \nu_1 = \frac{0.98-0.48}{0.98+0.94-0.68-0.48} = 0.6579 \quad ;$$

$$n_2 = \frac{0.88 \cdot (0.94-0.48)}{(0.98-0.68) \cdot (0.94-0.48) + 0.88 \cdot (0.98+0.94-0.68-0.48)} = 0.5017 \quad ;$$

$$c_2 = \frac{(0.98-0.68)\cdot(0.94-0.48)}{(0.98-0.68)\cdot(0.94-0.48)+0.88\cdot(0.98+0.94-0.68-0.48)} = 0.1710 \quad ;$$

$$\nu_2 = \frac{0.98\cdot(0.94-0.48)+0.88\cdot(0.98-0.94)}{(0.98-0.68)\cdot(0.94-0.48)+0.88\cdot(0.98+0.94-0.68-0.48)} = 0.6024 \quad ;$$

$$\gamma_2 = \frac{0.88\cdot(0.98+0.94-0.68-0.48)+0.68\cdot 0.48-0.98\cdot 0.94}{(0.98-0.68)\cdot(0.94-0.48)+0.88\cdot(0.98+0.94-0.68-0.48)} = 0.0917$$

Given this, we can now calculate the expected scoring probabilities for the three-strategy complete-information games, which are the following:

$$SP_1(CI) = \frac{0.98\cdot 0.94-0.68\cdot 0.48}{0.98+0.94-0.68-0.48} = 0.7826 \quad ;$$

$$SP_2(CI) = \frac{0.88\cdot[0.98\cdot(0.94-0.48)+0.94\cdot(0.98-0.68)]}{(0.98-0.68)\cdot(0.94-0.48)+0.88\cdot(0.98+0.94-0.68-0.48)} = 0.7993 \quad .$$

As we see, these results fulfill the rule found in section 3, under which "$SP_2(CI) > SP_1(CI)$".

If we now turn to the incomplete-information situation, we have two possible cases depending on the fact that θ is either greater than or less than 0.82895. When "$\theta > 0.82895$" (case A) [6], it will hold that:

$$n_1 = \frac{0.94-0.48}{0.98+0.94-0.68-0.48} = 0.6053 \quad ; \qquad \nu_A = \frac{0.98-0.48}{0.98+0.94-0.68-0.48} = 0.6579 \quad ;$$

$$n_2 = 0 \quad ; \qquad c_2 = 1 \quad ; \qquad \gamma_A = 0 \quad ;$$

whereas, if "$\theta < 0.82895$" (case B), it will hold that:

$$c_2 = \frac{(0.98-0.68)\cdot(0.94-0.48)}{(1-\theta)\cdot(0.98-0.68)\cdot(0.94-0.48)+0.88\cdot(0.98+0.94-0.68-0.48)} = \frac{0.1710}{1-\theta} \quad ;$$

$$\nu_B = 0.6024 \quad ; \qquad \gamma_B = 0.0917 \quad ; \qquad \theta\cdot n_1+(1-\theta)\cdot n_2 = 0.5017 \quad .$$

Besides, as we know from the results obtained in section 3, "$SP_1(IA) = SP_1(CI) = 0.7826$", "$SP_1(IB) = SP_2(IB) = SP_2(CI) = 0.7993$" and "$SP_2(IA) = \mu_2 = 0.88$". This implies that the average expected scoring probabilities under complete information and under the two incomplete-information cases are the following:

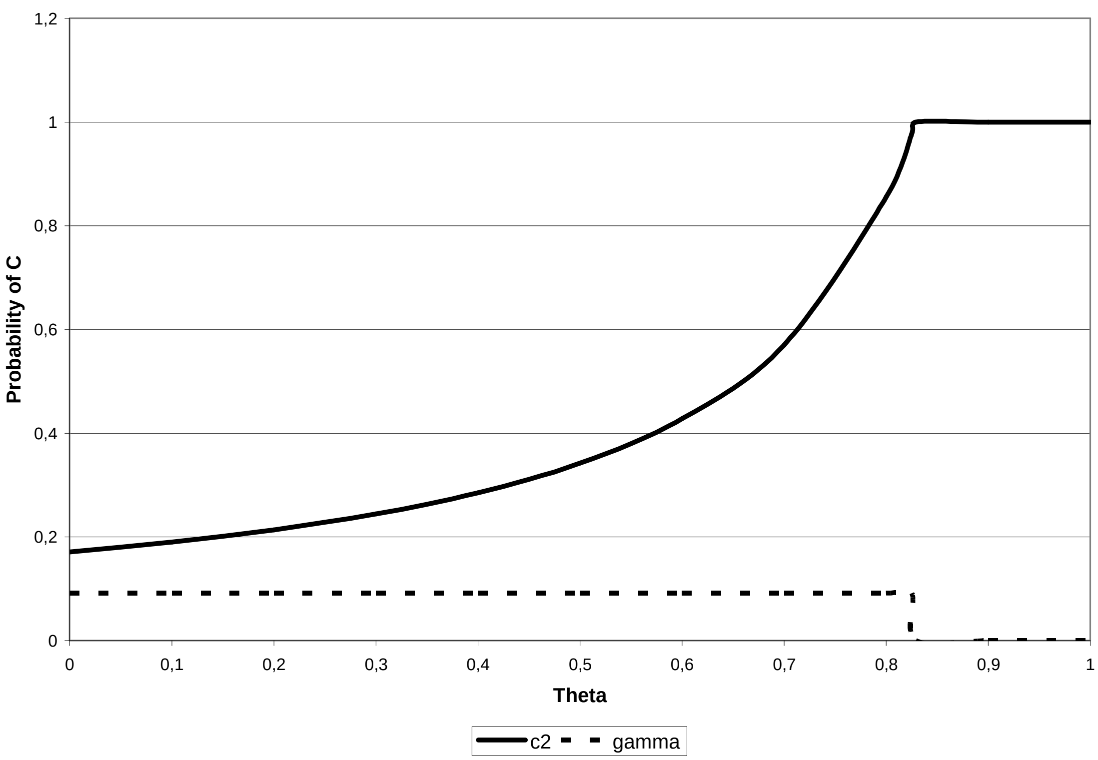
$$ASP(CI) = \theta\cdot 0.7826+(1-\theta)\cdot 0.7993 \quad ; \qquad ASP(IA) = \theta\cdot 0.7826+(1-\theta)\cdot 0.88 \quad ;$$

[6] This number comes from substituting the values of π_N, π_O, P_N, P_O and μ_2 into the formula found in propositions 6 and 7, which are the ones that define the range of values of θ for which case A and case B equilibria can occur.

$\mathrm{ASP(IB)} = 0.7993$.

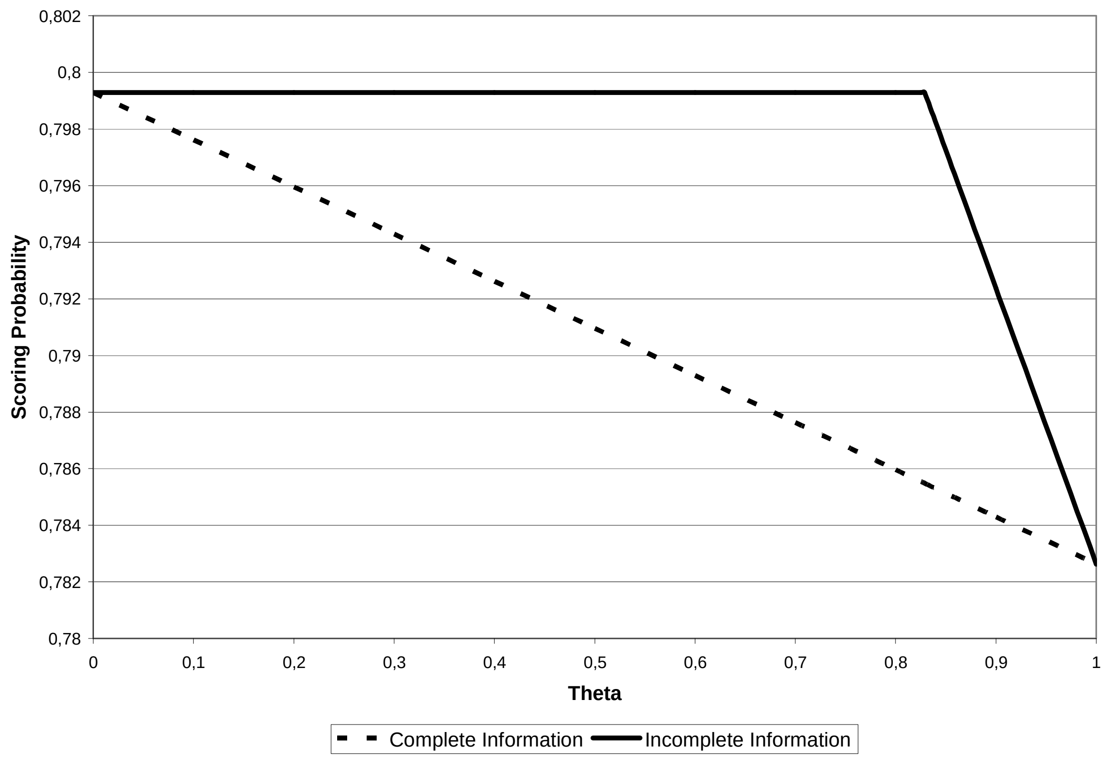
As we can see from the formulae, once again the incomplete-information cases produce some results that depend on the value of θ, that is, on the proportion of type 1 kickers that we have in the population under analysis. Figure 3 depicts the values of c_2 and γ_M that we obtain as equilibrium values for all possible levels of θ, and in that figure we can see that c_2 tends to its complete-information level when θ tends to zero, and becomes equal to one if "$\theta \geq 0.82895$". The value of γ_M, correspondingly, jumps from a value equal to the strategy chosen for a complete-information situation where the goalkeeper faces kicker 2 (γ_M = 0.0917) to a value equal to zero, and this also occurs when "$\theta \geq 0.82895$".

Figure 3: Equilibrium strategies under incomplete information (three-strategy model)



Correspondingly, figure 4 depicts the average scoring probability under complete and incomplete information for all possible levels of θ. In it we see that, unless "$\theta = 0$" or "$\theta = 1$", the average scoring probability is higher under incomplete information. We also see that, when θ increases, the average scoring probability under complete information decreases (since "$SP_1(CI) < SP_2(CI)$", and the average scoring probability is equal to "$\theta \cdot SP_1(CI) + (1-\theta) \cdot SP_2(CI)$"). The average scoring probability is also decreasing in θ under incomplete information if “θ > *0.82895*”, but for the levels of θ that are below that threshold it is constant and equal to the maximum possible average scoring probability (i.e., “*ASP(II) = 0.7993*”).

Figure 4: Average scoring probabilities (three-strategy model)



5. Empirical illustration

The numerical examples that we have built in the previous section, although based on parameter values estimated using real data, are not a true empirical illustration of our incomplete-information models, since they just try to find out the equilibrium values for those models under certain assumptions. In this section we will get closer to an empirical application of the models using some data reported in four empirical studies about the penalty-kick game, and we will try to see if the use of an incomplete-information approach can be helpful to improve the results of an equilibrium estimation. The exercise, however, will fall short of an actual empirical estimation of an incomplete-information model, basically because we will not use the original data which are the source of the empirical studies, but only some descriptive statistics that we will take as estimates of the underlying strategies and parameters of the model. The aim of this illustration, therefore, will not be to test an incomplete-information model but simply to show a possible approach to the problem of estimation of such a model in four particular situations.

The empirical studies that we will use as a source for our illustrations will be the already cited papers by Chiappori, Levitt and Groseclose (2002) and by Palacios-Huerta (2003), plus two more recent studies by Bar-Eli et al. (2007) and by Baumann, Friehe and Wedow (2011). The first two of those studies give strong evidence in favor of the reasonableness of the complete-information Nash equilibrium as a solution for the penalty-kick game[7], while the third one questions that evidence and points out a possible problem concerning the relatively small frequency that goalkeepers choose to stay in the center of the goal. The study by Baumann, Friehe and Wedow, finally, does not test the complete-information model but presupposes its validity, and tests the hypothesis that an increase in the quality of the kickers induces them to choose their natural side more often.

Table 3. Information from penalty-kick studies

Concept	CLG (2002)	PH (2003)	BEA (2007)	BFW (2011)
Average n	0,4488	0,4980	0,3917	0,4374
Average c	0,1721	0,0750	0,2867	0,1582
Average ν	0,5665	0,5310	0,4441	0,5435
Average γ	0,0240	0,0170	0,0629	0,0110
Implied π_N	0,9437	0,9648	1,0000	1,0000
Implied π_O	0,8992	0,9443	1,0000	1,0000
Implied μ	0,8418	0,8820	0,9304	0,6537
Implied P_N	0,6320	0,7120	0,7460	0,4922
Implied P_O	0,4400	0,5520	0,7040	0,3569
Average Scoring Rate	0,7490	0,8010	0,8530	0,7357

On table 3 we present a few data gathered from these four studies, which have been "translated" into our terminology of strategies (n, c, ν, γ) and scoring probabilities (π_N, π_O, μ, P_N, P_O)[8]. Of course, the numbers reported are not necessarily the actual strategies and probabilities but the average frequencies with which the players have chosen the different options (NS, OS and C) and the average scoring rates that occurred under the different combinations of those options. We also report the aggregate average scoring rates that correspond to the samples used in each of the studies. As the reader can imagine, "CLG"

[7] In another paper that we already cited in section 4 (Coloma, 2007), we have developed additional tests to check for the validity of the complete-information Nash equilibrium concept, but the data used are the same than the ones used by Chiappori, Levitt and Groseclose (2002).

[8] In three of the four cases the calculations were relatively easy, because the studies reported either the actual frequencies and rates or the actual number of shots needed to calculate those rates. For the case of the study by Baumann, Friehe and Wedow, conversely, we had to apply a very indirect method to detect the implied scoring rates in each of the strategy profiles.

means Chiappori, Levitt and Groseclose, "PH" means Palacios-Huerta, "BEA" means Bar-Eli et al., and "BFW" means Baumann, Friehe and Wedow.

Using the scoring rates that appear on table 3, it is relatively simple to calculate which would be the average Nash equilibrium strategies that players would have chosen if they had played in a complete-information environment. These are the ones predicted by equations 15, 16, 17 and 18, depending on the fulfillment of equation 14. To check this last condition it is necessary to calculate what we can call a "critical μ", that would be the maximum level of μ under which we can expect the occurrence of a restricted-randomization equilibrium. The first rows of table 4 show those complete-information equilibrium strategies implied by the four studies under analysis, together with the corresponding critical μ and the implied average scoring probability (ASP).

Table 4. Equilibrium results under complete and incomplete information

Concept	CLG (2002)	PH (2003)	BEA (2007)	BFW (2011)
Complete information				
Critical μ	0,7400	0,8030	0,8633	0,7163
Implied n	0,4881	0,5178	0,4692	0,5588
Implied c	0,1807	0,1484	0,1281	0,0000
Implied ν	0,5943	0,5936	0,5043	0,5588
Implied γ	0,0991	0,0762	0,0629	0,0000
Implied ASP	0,7584	0,8148	0,8719	0,7163
Incomplete information				
Estimated θ	0,9367	0,9725	0,8347	0,8418
Implied n_1	0,4791	0,5121	0,4692	0,5196
Implied n_2	0,0000	0,0000	0,0000	0,0000
Implied c_1	0,1162	0,0489	0,1455	0,0000
Implied c_2	1,0000	1,0000	1,0000	1,0000
Implied ν	0,6391	0,6296	0,5043	0,5588
Implied γ	0,0240	0,0170	0,0629	0,0000
Implied ASP	0,7578	0,8102	0,8719	0,7064

If we now compare the complete-information equilibrium results from table 4 with the information reported on table 3, we can see some striking similarities but also some important differences, which may cast some doubts about the ability of the complete-information model to explain the players' behavior. The implied average scoring probabilities, for example, are very similar to the actual average scoring rates in the four cases. The implied values of *c* for the CLG study, of *n* for the PH study, and of γ for the BEA and BFW studies are also extremely similar to the average values reported on table 3. Conversely, we can see that the calculated complete-information equilibrium predicts implied values for the parameters that are very different to the reported average values for the cases of the parameters n and c in both the BEA and BFW studies, for the parameters c, v and γ in the PH study, and also for the parameter γ in the CLG study. Moreover, the complete-information model predicts that the equilibrium in the BFW study should be one of restricted randomization (since the critical μ is larger than the parameter μ implied by the data), but we nevertheless observe a relatively large fraction of kicks that were actually shot to the center of the goal by the kickers in that sample.

Some of these divergences can be partially explained using a few easy incomplete-information assumptions like the ones made to calculate a new set of implied parameters (which are the ones that appear in the last rows of table 4). For the CLG case, for example, we have assumed that kickers are actually of two types: type-2 kickers strictly prefer to shoot to the center of the goal, while type-1 kickers choose a mixed strategy that combines NS, OS and C with positive probability. To match the data on the observed choices of NS, we had to assume a certain distribution of the types (the "estimated θ"), and based on that we also estimated a certain value for the implied parameter c_1. The parameter γ, conversely, was supposed to be equal to the observed average value for that parameter, while v was estimated as the value that made type-1 kickers indifferent between choosing NS, OS and C.

The same methodology for defining the two types of kickers were used to match the data reported in the PH and BEA studies. For the BFW study, however, we had to use a different approach to conciliate the prediction of the complete-information model (that on average it was not optimal for the kickers to choose C) with the data that show that 15.82% of the kicks were actually shot to the center of the goal. In order to solve that puzzle, we assumed that in this case type-1 kickers were players who never chose C and type-2 kickers were players who always shot to the center of the goal[9]. These assumptions allowed us to estimate a certain value for θ, but obliged us to assume that the implied value for γ was equal to zero. This last feature does not exactly match the data (since the average γ in the BFW study is 0.011), but it helps us to explain how it is possible that there is such a large fraction of kickers that choose C in equilibrium while almost no goalkeeper is willing to stay in the center of the goal.

6. Final remarks

The main conclusions of this paper have to do with the idea that, in some cases, the outcomes of a situation in which a soccer goalkeeper faces a kicker at a penalty kick can be

[9] Of course, this implies assuming that type-2 kickers are players whose scoring probabilities when shooting NS or OS are completely different (lower) than the ones associated to type-1 kickers. This lower scoring probabilities are never observed, since type-2 kickers always choose C instead of NS or OS.

better explained as the result of an incomplete-information game. In those cases, the relevant solution concept is no longer the mixed-strategy Nash equilibrium of the game but the corresponding Bayesian equilibrium, since at least one of the players (e.g., the goalkeeper) is facing uncertainty about his opponent's type.

In the simplified models that we presented, we see that, under incomplete information, the typical situation is that at least one of the player's types (i.e., one of the kickers) chooses a pure strategy instead of a mixed strategy. This choice, which is almost impossible in equilibrium under complete information, arises because that type of kicker is actually responding to a strategy that the goalkeeper has designed for a different type of opponent. Being unable to distinguish among the different types, the goalkeeper has to play the same strategy against every opponent, and this is why some types of kickers may prefer a pure strategy. When we mix the strategies played by the different kickers, however, we end up with a sort of "average kicker strategy" with different probabilities for the available pure strategies, and this average strategy has to be such that the goalkeeper is indifferent between playing the pure strategies that he mixes when he chooses his own best response to the "expected kicker".

In the three-strategy model presented in section 3, we also have cases in which one of the kicker's types plays a "restricted mixed strategy" (e.g., one that randomizes between NS and OS) while the other type plays a "full mixed strategy" (i.e., one that randomizes among NS, OS and C). We can also end up in a situation in which one of the kicker's types plays a restricted mixed strategy and the other one plays a pure strategy, and the goalkeeper chooses a restricted mixed strategy himself (which is the best response to the kicker who plays the restricted mixed strategy). This last case produces the apparently paradoxical situation that, in equilibrium, the goalkeeper never chooses the center of the goal while one of the kicker's type always shoots to that place.

The relative lack of information that the goalkeeper faces in a situation of incomplete information makes that the average scoring probability is higher than under a situation of complete information, which is equivalent to say that, on average, the kicker is better off under incomplete information and the goalkeeper is worse off. This feature can therefore be used to find the "value of information" in this game. As goalkeepers' payoffs are the complements of the scoring probabilities, the value of knowing the true characteristics of a kicker can be measured as the difference between the expected scoring probability under complete and incomplete information. This difference is smaller if we are in a situation in which uncertainty is small (i.e., when the parameter θ, which measures the distribution of the kicker's types, is very close to zero or to one) and becomes larger when we approach the level of θ where the

Bayesian equilibrium of the game changes from case A to case B. The difference will also be larger if the different types of kickers are "more different" among themselves.

Another virtue that the incomplete-information approach could have is to solve some puzzles that the empirical literature on penalty kicks has discovered. Indeed, we have seen on section 5 that the Nash equilibrium concept performs quite well to explain some phenomena observed in a number of empirical studies about penalty kicks, but that some other features are hard to explain using a complete-information approach. This is particularly true for the relatively widespread fact that concerns shots to the center of the goal, which are typically more common than what a complete-information Nash equilibrium predicts. This phenomenon has been analyzed by Bar-Eli et al. (2007) as a weakness of the game-theoretic approach to penalty kicks, and it has been explained by those authors using an alternative approach (called "norm theory") derived from psychological economics. The essence of that approach is that goalkeepers are not actually minimizing an expected scoring probability but following a social norm that prescribes a certain action (jumping to the right or to the left) instead of a situation of "inaction" (i.e., staying in the center of the goal). If the "social penalty" for choosing C when the kicker chooses NS or OS is higher than the one received for choosing NS or OS when the kicker chooses C, then a goalkeeper may prefer not to choose C in any situation, although he knows that he can reduce the expected scoring probability by choosing the center of the goal instead of jumping to one of its sides.

By introducing incomplete information, however, the situation described in the previous paragraph can be explained as the result of a game-theoretic equilibrium. Without recurring to psychological arguments, we have seen that it can be optimal for a goalkeeper to randomize between NS and OS although he knows that a group of kickers will always choose C, provided that such a group of kickers is relatively small. We have also seen that it is possible to think of certain Bayesian equilibrium solutions in which the goalkeeper randomizes among NS, OS and C, and the different types of kickers choose more restricted mixes (e.g., between NS and OS) or even pure strategies.

The main analytical problem of introducing incomplete information into the penalty-kick game may perhaps be its extreme capacity to match the data. Indeed, if we build a game of incomplete information that postulates more than two types of players and we arbitrarily use different probabilities for those types, then we could probably explain any dataset on penalty kicks as a result of a particular Bayesian equilibrium. If that is the case, then many of the empirical tests that the penalty-kick game-theoretic literature has designed could become useless, since it would actually be impossible to distinguish between a Bayesian equilibrium and a situation in which the players are not choosing their strategies rationally.

We nevertheless believe that the Bayesian equilibrium concept also opens the door for new possible empirical estimations of the penalty-kick game, especially in cases in which it is not clear whether the goalkeepers know their opponents' types. This is particularly true for situations in which the expected incomplete-information solution is markedly different from the expected complete-information solution, and especially when we can somehow divide a sample of penalty kicks into different types of kickers[10]. In those cases, it could be possible to contrast the predictions of the complete-information Nash equilibrium concept with the ones of the incomplete-information Bayesian equilibrium concept, and also with other alternative concepts that are foreign to the game-theoretic approach.

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[10] This division may rely on phenomena that are not *ex-ante* observed by goalkeepers but are *ex-post* observed by the analyst (e.g., if a kicker never chooses a certain action, the speed of the shots that he kicks, etc.).