UNIVERSITY OF CEMA
Working Paper Series, number 239
July 2003

ON THE EXTENT OF ARBITRAGE CONSTRAINTS

WITHIN TRANSACTION ALGEBRAS

(A NON-STANDARD APPROACH)

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ABSTRACT

Although the standard trading arbitrage model provides with simple settings and adjustment mechanisms so as to take profit whenever an arbitrage opportunity comes up, empirical evidence has been piling up showing that this point of view suffers from many downsides, leaving distinctive issues unresolved. By the same token, similar shortcoming prevent the standard financial arbitrage model from being functional to real markets environments. To overcome such drawbacks, this paper sets forth a new approach that is grounded on transactional algebras, which shapes the arbitrage gaps of return within institutional settings, to give account of market microstructure features and enlarged transaction costs.

JEL : G10, G12, G14

Key words: Arbitrage, Transaction Costs, Residual Information Sets, Differential Rates of Return, Arbitrage Gaps

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In its purest form, arbitrage requires no capital and is risk free. By simultaneous selling and purchasing identical securities at favorably different prices, the arbitrageur captures an immediate payoff with no up-front capital. Unfortunately, pure arbitrage exists only in perfect capital market. Real world arbitrage is capital intensive and risky. Market frictions and imperfect information make arbitrage costly, thereby limiting its effectiveness in keeping prices at their fundamental values.
M. Mitchell, T. Pulvino, E. Stafford. (2001)." Limited Arbitrage in Equity Markets".

INTRODUCTION

For centuries, arbitrage has been a tool of the trade for businessmen and investors. Since the very inception of commerce, traders have been engaging themselves in the discovery and profiting from arbitrage opportunities, by buying a good at any time and place, reselling it later at the same or another place, and reaping a monetary reward after rounding off the transaction. Therefore, the rationale for any traditional arbitrage consists of selling what is relatively overpriced and buying what is relatively underpriced.

As we are interested mainly in financial arbitrage, it can be worthy of interest to recall what the Encyclopedia Britannica said in its 1875 edition about arbitrage in the capital markets, because current usage kept its remarkable semantic aside for decades till the 70s when both the transaction costs and market microstructure approaches made a definite inroad in the academic discussion. This was the explanation of arbitrage given at that time:

By arbitrage is meant "*a traffic consisting of the purchase (or sale) on one Stock Exchange and simultaneous or nearly simultaneous resale (or repurchase) on another (...) at a difference in price sufficient to cover the cost of transmission, commission, interest and insurance, and leave an adequate profit (...) to be divided by the operators at both ends.* "

Although the 1910 edition still expanded on the role of risk in arbitrage and the superior information on the arbitrageur's side, as Weisweiller (1987) remarked, the sound semantic exhibited in the 1875 edition was surprisingly played down in later editions. For instance, looking up in the 1995 printing of the 15th edition, arbitrage is defined there as merely involving the purchase of foreign exchange, gold, financial securities, or commodities in one market and their almost simultaneous sale in another market in order to profit from price differentials existing between the markets, without adding up to any further qualification, even neglecting the basic features involved in temporal arbitrage.

It is for any arbitrage opportunity to trigger off a process by which the payoff will be eventually narrowing down till reaching a final stage when the benefit eventually fades away. At this point, it is said that a zero-arbitrage state has been reached. The customary example is provided by borrowing and lending costlessly at two different rates of interest, grabbing a monetary reward whenever playing with different currencies.

As from the 70s, zero-arbitrage settings have been thoroughly studied (Varian, 1987) mainly because they provide neutral frameworks of valuation for financial assets, either primary securities or derivatives (Ross, 1978) and also because the underlying mathematics runs more straightforwardly. The New Palgrave Dictionary of Money and Finance (1992) has outlined this perspective, when defining the arbitrage opportunity as a strategy with the following features:

a) Positive payoff in some contingencies
b) No possibility of a negative payoff
c) No actual investment

Absence of arbitrage provides environments that seem to be "more primitive than equilibrium ones, since only relatively few rational agents are needed to bid away arbitrage opportunities, even in the presence of a sea of agents driven by animal spirits " (Dybvig and Ross, 1992).

The idea of zero-arbitrage presents itself, however, like a rather mechanistic view about economic or financial processes, as Ellerman (1984) thoroughly showed. The basic arbitrage theorem dates from Cournot: "there exists a system of absolute prices for commodities such that the exchange rates (or relative prices) are price ratios if and only if the exchange rates are arbitrage-free." Ellerman puts forth that such format amounts to the multiplicative version of Kirchhoff's Voltage Law in electrical circuit theory.

It seems hardly surprising that in the last twenty years painstaking efforts have been made to overcome this constraining frame of mind, although keeping the zero-arbitrage approach safe since it may provide with useful benchmarks for valuation. It is within a non-standard pathway to arbitrage that this paper will labor over the following issues:

Section 1 sets forth a convenient format for any standard trading arbitrage process. Next, it addresses empirical evidence of shortcomings either in the standard arbitrage or the so-called law of one price. The section closes with the underlying dynamics of the standard arbitrage. Similar targets are pursued in section 2, but this time addressing only financial arbitrage processes.

The concept of transactional algebra is the issue developed in section 3, encompassing an institutional setting, an explicit transaction costs rate and residual information sets. Groundwork is done with sets structures, differential rates, residual information sets, so as to smoothly proceed to the notion of transactional algebras. This approach leads to differential transaction costs rates that measures the cost of running a transactional algebra within an arbitrage process.

Finally, it is for section 4 to expand on conditions under which the arbitrage can take place, firstly by designing an arbitrage return net of transaction costs and, secondly, by establishing boundary constraints for making the arbitrage profitable. This section concludes with practical implications and provides a fully worked-out numerical illustration.

1. THE STANDARD TRADING ARBITRAGE MODEL

By and large, trading arbitrage could be summarized as "buy low and sell high". However, we can also perform arbitrage by selling certain asset at an expected moment and repurchasing it later at a lower price (what is termed as temporal arbitrage). If we owned the asset, that would mean opening a short position. If we did not own the asset, we would open a short-selling position. Limitless short selling comes down to a crucial assumption in many Financial Economics models, in spite of being tightly limited or banned outright around most real markets. Therefore, and to also take into account the many varieties in which arbitrage usually takes place, it will be functional to set up a broader definition than the one currently used.

Henceforth, " l " stands for long position, and " s " for short position. In general, most of arbitrages turning out in the markets fall into two wide-ranging types: those that start with a long position and end up with a short one (long-short type), and those that start with a short position and end up with a long one (short-long type), denoting their payoff functions ***Π(long-short)*** and ***Π(short-long)***, respectively.

Definition 1

Trading Arbitrage *is a decision making process whose main features are:*

i. *the trade of certain merchandise or service "g_1", at an expected moment "t_1", in a certain market "m_1", at the value* $\boldsymbol{V(g_1; m_1; t_1)}$ *;*

ii. *the trade of certain merchandise or service "g_2", at an expected moment "t_2", in a certain market "m_2", at the value* $\boldsymbol{V(g_2; m_2; t_2)}$, with $t_1 \leq t_2$ *;*

iii. *making a sure profit from round-off transactions, either the long-short or the short-long types, that is to say, the payoff functions* $\Pi(\,.\,)$ *are positive*

$$\begin{cases} \Pi(\text{long-short}) = V(g_2; m_2; t_2; s) - V(g_1; m_1; t_1; l) > 0 \\ \Pi(\text{short-long}) = V(g_1; m_1; t_1; s) - V(g_2; m_2; t_2; l) > 0 \end{cases}$$

As far as the payoff functions are profitable, we say that an arbitrage opportunity can be met with success.

Remarks

Vectorial notation comes in handy here because we can include or leave out as many components as needed to keep the functionals $\boldsymbol{V(\,.\,)}$ *precisely defined. For instance, in* $\boldsymbol{V(g_1; m_1; t_1)}$ *we are interested in the asset, the market and the date of trade. Instead, when we use* $\boldsymbol{V(g_1; m_1; t_1; l)}$ *the stress shifts to the long position as opened at that moment and market.*

Definition 1 allows for trading not only with the same economic good at different dates through different markets, but with close substitutes as well. In contrast, the standard model of arbitrage has usually been predicated on perfect substitutes only.

In this section we deal with non financial merchandise and services, leaving for section 2 the extension to financial arbitrage.

The conventional arbitrage grants a sure profit. When $t_1 = t_2$, both asset values are known with certainty. If $t_1 < t_2$, however, the investor should lock in the future value; otherwise, the round-off transaction becomes risky and much closer to speculation than to pure arbitrage. By means of derivatives markets (mainly options and futures) investors can lock in prices whenever they engage themselves in temporal arbitrage.

A cautious qualification seems due here. In futures and option market exchanges only standard and limited varieties of trade are allowed. This means that hedging falls short of being perfect most of the time. It is for the over the counter forward markets to tailor derivatives by request, customizing the investor's needs. It must be borne in mind that the bigger the players are, the most they can draw from derivatives hedging.

Three examples can highlight the broad range conveyed by definition 1.

Example 1 Temporal Arbitrage: Purchasing cheaply, selling expensively later

A merchant finds out that by buying cranes with bottles of vintage wines directly from distinguished wine-makers to resell them later in leading downtown hotels and restaurants, this could give him a profit advantage for certain. He is taking advantage of some experts who claim the wine will improve their quality in a couple of years. In the framework of definition 1:

$$g_1 = g_2 \qquad t_1 < t_2 \qquad m_1 = m_2$$

$$\Pi(\text{long-short}) = V(t_2; s) - V(t_1; l) > 0$$

Example 2 Temporal Arbitrage: Selling expensively, repurchasing cheaply later

The same merchant overinvested last year in premium red wines but, at present, economic variables makes a turnabout in prices for the year ahead. He forecasts the worst drop in prices in six months later, with another semester of steady low prices. He sells the whole of his inventory and repurchases six months later.

$$g_1 = g_2 \qquad t_1 < t_2 \qquad m_1 = m_2$$

$$\Pi(\text{short-long}) = V(t_1; s) - V(t_2; l) > 0$$

Example 3 Spatial Arbitrage: Purchasing cheaper, selling expensive at another place

A merchant finds out that buying cranes with bottles of vintage wines directly from certain wine-makers to immediately resell them to neighbor countries wine stores, it will give him a distinctive profit pattern. In the framework of definition 1:

$$g_1 = g_2 \qquad t_1 = t_2 \qquad m_1 \neq m_2$$

$$\Pi(\text{long-short}) = V(m_2; t_2; s) - V(m_1; t_1; l) > 0$$

for Π(long-short), both expressions in domestic currency

Examples 1,2, and 3 could also be framed into the auction business setting, like those performed by the wine divisions at Sotheby's or Christie's in which the law of one price seems repealed most of the times. On this account, Ashenfelter (1989) proves to be not only amusing but interesting as well.

1.1. THE LAW OF ONE PRICE

Strongly related to the idea of zero-arbitrage, it seems to be the Law of One Price (LOP). The classical definition pertains to International Finance and states that, abstracting from transportation costs, obstacles to trade, and information costs, the price of any good $\mathbf{g}_k$ will be the same in different locations:

$$\mathbf{p(g_k, t, m_1) = p(g_k, t, m_2) . e(t, \$m_1 / \$m_2)}$$

where $\mathbf{e(t, \$m_1 / \$m_2)}$ is the exchange rate (home currency price of foreign currency); $\mathbf{p(g_k, t, m_1)}$ is the domestic currency price of good $\mathbf{g}_k$, and $\mathbf{p(g_k, t, m_2)}$ is the foreign currency price of the same good. In general, what the LOP states is that economically identical goods must have identical prices, independently of investors' preferences.

Another meaning attached to the Law of One Price has the following format: two perfect substitutes must trade at the same price. For example, two shares of stock in the same company, under the same terms of issuance, cannot be priced differently. Otherwise, an arbitrage opportunity should arise, by which we may buy the cheaper share and sell the more expensive one.

Remark

Sometimes, the LOP is also called "law of indifference" in a context of definition that goes back to Jevons: one and the same good or commodity should sell, at the same time, at only one price, provided that transaction and transportation costs are negligible.

The LOP assumes perfectly competitive markets, and perfect substitutability between domestic and foreign goods. We see that, in order for the LOP to hold true, the assets involved should be identical and relatively mispriced. In fact, the Law gives only one particular environment where arbitrage processes might evolve. However, product differentiation, trade barriers, information costs, delivery lags, distribution and insurance features, bring about heterogeneous tradables, to the extent that the Law of One Price does not seem to hold in the short term even for homogeneous tradables (see section 1.2 on the empirical evidence of shortcomings in the Law of One Price.

The problem of market integration is one of great concern when measuring mispricing of assets and violations of the LOP variants. On this account, Chen and Knez (1995) set forth two alternative approaches to market integration:

- Firstly, they remarked that two markets cannot be integrated in any sense if it is possible to construct two portfolios, one from each market, that have identical payoffs but different prices. Hence, two markets are said to be perfectly integrated it the Law of One Price holds across them.
- Secondly, evidence shows that two markets, although regarded as perfectly integrated, can nurture arbitrage opportunities between them. In this case, Chen and Knez suggest to assess the extent by which a zero-arbitrage setting is violated as a measure of non integration.

A third variant of the LOP is the Purchase Power Parity (PPP): once converted to a common currency, national price levels should be equal. The usual explanation conveys the idea that market arbitrage enforces broad parity in prices across a sufficient range of individual goods, and then there should also be a high correlation in aggregate price levels. There is consensus on a couple of basic facts:

- In the long run, real exchange rates only slowly converge towards PPP.
- Short-term deviations from PPP are large and volatile.

It is this contradiction that gives birth to the so called “PPP puzzle”, because there is sensible empirical evidence for both statements to be regarded as true at the same time (Rogoff ,1996).

As a theory of equilibrium, PPP must be grounded on an adjustment mechanism. When the commodities are identical, the theory is simply that of spatial arbitrage. If the goods were not strictly identical, world trade would be expected to exhibit a wide degree of substitution.

1.2. SHORTCOMINGS IN THE STANDARD TRADING ARBITRAGE MODEL : EMPIRICAL EVIDENCE

In the last thirty years, evidence against the standard arbitrage model has been gathering to such an extent that unremitting efforts are being made to overcome the strictures on the traditional point of view. We are going to survey some influential contributions.

a) It was Isard (1977) one of the first economists to study persistent violations of the law of one price in real markets. In particular, his research focused on products manufactured in different countries, and deviations that arise from restrictions in international trade. Exchange rates influence in relative prices and these changes in prices cannot be regarded as transient. Therefore, products become differentiated instead of being close substitutes of each other.

b) As Froot, Kim and Rogoff (1995) put it “one of the most striking empirical regularities in international finance is the volatility and persistence of deviations from the law of one price across relatively homogeneous classes of goods.” They found these main features:

- Goods market arbitrage is quick for precious metals but slow for most goods, with half-lives for price deviations exceeding one year.
- Volatility of deviations from the law of one price has been no larger in the twentieth century than in the fourteenth.
- Regardless of plagues, wars and trends, the volatility of the law of one price deviations is both remarkably high (20 % or more per year) and stable over time for most commodities.

c) Knetter (1997) studied the behavior of newsstand prices of *The Economist* magazine in eight national markets. Although substantial variations in markups across markets are related to exchange rate fluctuations, he found that a substantial explanation for systematic departures from price equalizations could be traced to:

- Intentional price discrimination across the United States, the United Kingdom and Continental Europe and Scandinavia.

- Differences in demand elasticities can plausibly be attributed to differences in preferences and the set of competing products across markets.
- Segmentation of the markets is facilitated by the time-sensitive nature of the product that makes arbitrage very costly.

d) In a survey led by Engel and Rogers (1999), 29 US cities were surveyed, from December 86 to June 96, taking advantage of indices for 43 different goods and an overall CPI for each city as well. Deviations from the proportional law of one price (PLOP) across US cities can be traced to the following facts and explanations:

- Distance between locations: most of it can be brought about by transportation costs.
- Volatility of nominal prices: because of sticky nominal prices.
- Deviations are larger for traded-goods: on the grounds of greater price stickiness from non-traded goods.

e) An earlier paper by Engel and Rogers (1994) labored on transportation costs by relating measures of the spread or variance of the price of similar goods to the distance between markets, for nine Canadian cities and fourteen cities in the United States. The authors found out that border matters when we try to explain the failure of the law of one price.

- The price of a consumer good may be sticky in terms of the currency of the country in which that good is sold.
- The cross-border prices can fluctuate along with the exchange rate, although the within-country prices remain stable.
- Market segmentation may reinforce price stickiness.

They remarked that the price of the final good is inclusive of the services that the good brings to the market (advertising, quality control, reputation, retailing, for instance). Interregional variation in prices of a good translates variation in the costs of this marketing service, and prevents arbitrage from taking place.

f) A recent piece of empirical evidence on deviations from the law of one price is provided by Haskel and Wolf (2001). The authors used retail transaction costs for a multinational retailer to examine the extent and permanence of such deviations.

- The sample consisted in 100 identical goods sold in twenty-five countries by IKEA, a Swedish furniture retailer.
- Haskel and Wolf used absolute prices to avoid problems met by former researchers when dealing with price indices.
- They found significant common currency price divergences across countries for a given product and across products for a given country. The distribution properties of the price divergences suggest that they cannot be attributed to differences in local costs, tariffs or taxes.
- First conclusion: sizable differences in markups led to a failure of effective arbitrage.
- Second conclusion: violations from the law of one price are pervasive features of individual goods prices. (Typical deviations of twenty to fifty percent, for identical products).

g) It was for Rogoff (1996) to come up with this question: " How is it possible that goods markets arbitrage do not force closer convergence of international prices? ". He displays three main reasons: transportation costs, information costs and tariffs.

Furthermore, he firstly hints at some markets where arbitrage is difficult or impossible: automobiles and electronics; and secondly, at the example of the monopolistic firm that refuses to provide warranty service in one country for goods purchased in another. All these features drive a wedge between prices and negatively impinge upon arbitrage.

h) Asplind and Friberg (2001) researched prices from Scandinavian duty-free outlets where each product (at the same location) has price tags in at least two currencies. It could be thought that the LOP holds well in this setting, but the potential for arbitrage arises due to exchange rates daily fluctuations not followed by continuously price adjustments. Small deviations from LOP may persist, but large deviations are quickly adjusted, and so as to explain this the author expand on costly arbitrage and fixed costs of adjusting nominal prices.

1.3. THE STANDARD TRADING ARBITRAGE DYNAMICS

Any arbitrage opportunity stems from a tractable mispricing between identical or substitute goods, triggering off a process of price adjustment. In the standard arbitrage model the dynamics of such adjustment is almost instantaneous, but such process seems almost unattainable in real markets on the grounds of evidence surveyed in section 1.2.

Arbitrage makes sense only to the extent of being profitable, and the adjustment comes to an end when either of the following relationships is met:

$$\left\{\begin{array}{lcl} \Pi(\text{long-short}) & = & V(g_2; m_2; t_2; s) - V(g_1; m_1; t_1; l) \rightarrow 0 \\ \Pi(\text{short-long}) & = & V(g_1; m_1; t_1; s) - V(g_2; m_2; t_2; l) \rightarrow 0 \end{array}\right. \tag{1}$$

On the other hand, mispricing is a relative concept that means at least two things:

a) A discrepancy has aroused between current trading prices, as shown in definition 1.

b) A discrepancy has aroused between both the current trading price and either its long or short counterpart, in contrast to a benchmark value

$$\mathbf{W(t)}$$

In the conventional setting of arbitrage, it has been customary to resort to equilibrium values that stand for **W(t)**. For the last thirty years, and mainly in the field of Financial Economics, valuation models provide with benchmark values as proxies of unobserved equilibrium prices.

Remarks

Even when we attempt to arbitrage the same economic good, benchmark values become essential, most of all when temporal arbitraging is at stake.

It is in spatial arbitrage where the benchmark value could be felt as useless. This belief, however, is misplaced. If we can buy good g_1 at [V] dollars, in market m_1, and sell it up to $[V + \alpha ; \alpha > 0]$ dollars in another market m_2, (it may hold that $m_1 = m_2$), the adjustment process will shift the transaction value at a level W(t) not to be contested by market forces and fundamentals, although the adjustment might not come to zero as predicated by the standard model.

Following Mandelbrot (1971), an arbitrage environment claims for a pair of dynamical systems: the first one, to account for the benchmark value evolution

$$\{ W(t) : t \geq t_0 \} \tag{2}$$

and the second coming as a vectorial dynamical system carrying the market price evolution of any asset g_1 and that of its likely substitute g_2

$$\{ V(g_2; m_2; t), V(g_1; m_1; t) : \quad t \geq t_0 \} \tag{3}$$

Remarks:

In many contexts of application, it holds that $g_1 = g_2$.

Mandelbrot actually refers to value or price series, within a stochastic approach. By the side of this pathway, and after 30 years of academic effort at researching in non linear dynamics, scholars seem prone to deal with this more general framework,, which allows deterministic environments, either in continuous or discrete settings. (Martelli, 1999; Barkley Rosser, 2000; Apreda, 1999; Devaney, 1989).

In Exhibit 1 we can follow the likely long-short and short-long patterns of arbitrage opportunities. We want to highlight the following features concerning the long-short side of the arbitrage:

- At date $\mathbf{t = t_1}$, an arbitrage opportunity is discovered, while at date $\mathbf{t = m}$ we signal the time when such an arbitrage opportunity fades away.
- Imagine that the investor founds out that good $\mathbf{g_1}$ is undervalued at $\mathbf{t = t_1}$ and he is certain that there will be later an adjustment of its price towards a fundamental or equilibrium value by the date $\mathbf{t = t_2}$. Hence, the condition

  $$V(g_1; m_1; t_1; l) \quad < \; E[\, V(g_2; m_2; t_2; s)\,]$$

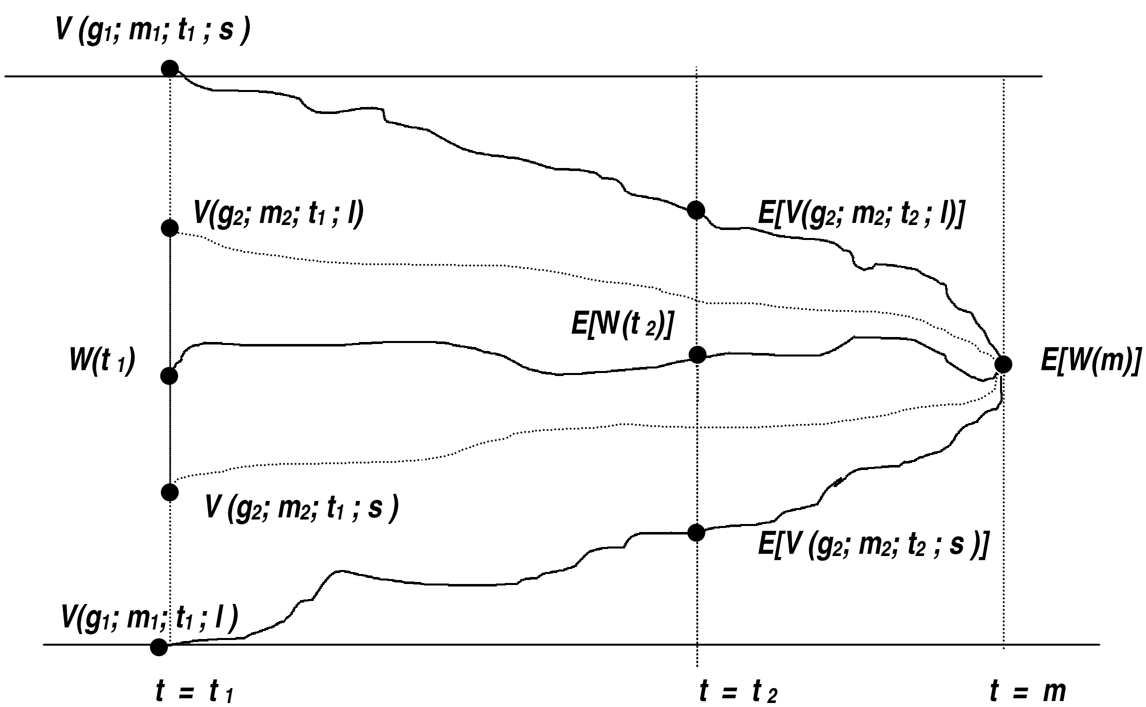
  triggers off a temporal arbitrage opportunity. Although we highlight here the expectations operator E[.] , we are going to drop it whenever were implied by context.
- If the investor founds out that $\mathbf{g_1}$ is undervalued at $\mathbf{t = t_1}$, whereas the same good or a perfect substitute could be sold at the same date, in the same or other market, at a higher price, then he will intend to buy the undervalued good and to sell the overvalued one. Hence, the condition

$$V(g_1; m_1; t_1; l) \quad < \quad E[\,V(g_2; m_2; t_1; s)\,]$$

triggers off a spatial arbitrage opportunity.

- By the same token, we could deal with the short-long patterns, as depicted in the upper side of Exhibit 1.

Exhibit 1: <u>STANDARD TRADING ARBITRAGE DYNAMICS</u>



<u>References</u>

Horizontal axis:	*time history for the monetary valuation of the asset*
Vertical axis:	*monetary valuation of the asset*
W(t):	*monetary valuation of the benchmark along its time history*
t_1:	*date at which an arbitrage opportunity is found out*
t_2:	*date at which the transaction is rounded off (temporal arbitrage)*
t_1:	*date at which the transaction is rounded off (spatial arbitrage)*
t_m:	*date at which the price adjustment comes to an end*

Bearing in mind the qualifications discussed above, we are to expand on the adjustment conditions when a benchmark is included. Conventional analysis relies on assuming that when valuing the asset and its substitutes it must hold

$$W(g) = W(g_1) = W(g_2) \tag{4}$$

that is to say, either with equilibrium prices or valuation models outputs, as long as arbitrage opportunities are alive, both the good and its perfect substitute have the same benchmark value drawn out from a valuation model. Therefore, two alternative dynamics follow, contingent upon the type of arbitrage process under way: long-short or short-long.

a) Long-short arbitrage

$$\left\{ \begin{array}{l} \left| V(g_2; m_2; t_2; s) - W(g) - \{ V(g_1; m_1; t_1; l) - W(g) \} \right| \leq \\ \left| V(g_2; m_2; t_2; s) - W(g) \right| + \left| V(g_1; m_1; t_1; l) - W(g) \right| \rightarrow 0 \end{array} \right. \tag{5}$$

Also in Exhibit 1, there are two plausible patterns that hold with immediate arbitrage (dotted lines going from $t = t_1$ till $t = m$). This sort of arbitrage arises in spatial spreads environments, or even with the same market. In this case, the dynamics differs a little from the one conveyed in (5).

$$\left\{ \begin{array}{l} \left| V(g_2; m_2; t_1; s) - W(g) - \{ V(g_1; m_1; t_1; l) - W(g) \} \right| \leq \\ \left| V(g_2; m_2; t_1; s) - W(g) \right| + \left| V(g_1; m_1; t_1; l) - W(g) \right| \rightarrow 0 \end{array} \right. \tag{6}$$

b) Short-long arbitrage

By the same procedure, if we dealt with the short-long type of standard arbitrage, we would attain:

$$\left\{ \begin{array}{l} \left| V(g_1; m_1; t_1; s) - W(g) - \{ V(g_2; m_2; t_2; l) - W(g) \} \right| \leq \\ \left| V(g_1; m_1; t_1; s) - W(g) \right| + \left| V(g_2; m_2; t_2; l) - W(g) \right| \rightarrow 0 \end{array} \right. \tag{7}$$

This type of arbitrage carries on similar oversimplifications as those we have already met when dealing with the long-short position.

2. THE STANDARD FINANCIAL ARBITRAGE MODEL

Within an extreme framework, pure theory of arbitrage means that equilibrium comes out of the fact that no economic agent is able to make profit without risking at least some capital (no free lunches). The underlying assumptions are those usually laid down on perfect markets. A noteworthy feature of this theory (as noticed by Garman, 1979) lies on the fact that explains the behavior of capital markets without resorting to the economic agents preferences. From the 1970s, arbitrage has been widely tracked down and studied in the financial markets, giving rise to many developments in financial engineering. Although it keeps the basic features of the trading arbitrage, it also adds some of its own, making advisable to put forth a specific format of definition.

Definition 2

Financial Arbitrage *is a decision making process whose main features are:*

i. *the trade of a financial asset "g_1", at an expected moment "t_1", in a certain market "m_1", at the value* $\boldsymbol{V(g_1; m_1; t_1)}$ *;*

ii. *the trade of a financial asset "g_2", at an expected moment "t_2", in a certain market "m_2", at the value* $\boldsymbol{V(g_2; m_2; t_2)}$, with $\boldsymbol{t_1 \leq t_2}$ *;*

iii. *making a sure profit from round-off transactions, either the long-short or the short-long types, that is to say, the payoff functions* $\Pi(\,.\,)$ *are positive:*

$$\begin{cases} \Pi(\text{long-short}) = V(g_2; m_2; t_2; s) - V(g_1; m_1; t_1; l) > 0 \\ \Pi(\text{short-long}) = V(g_1; m_1; t_1; s) - V(g_2; m_2; t_2; l) > 0 \end{cases}$$

iv. *no investment is required for rounding off both transactions;*

v. *risks of rounding off both transactions are null.*

An equivalent statement for condition iii will become useful to definition 3:

$$\begin{cases} \Pi(\text{long-short}) = V(g_2; m_2; t_2; s) \,/\, V(g_1; m_1; t_1; l) > 1 \\ \Pi(\text{short-long}) = V(g_1; m_1; t_1; s) \,/\, V(g_2; m_2; t_2; l) > 1 \end{cases}$$

Whereas trading arbitrage requires of an asset (or two close substitutes) to be mispriced, financial arbitrage takes place within a much more encompassing setting. In fact, to what extent can we regard two financial assets as close substitutes? A sensible approach qualifies two assets or portfolios to be substitutes when they fulfill the following features:

- their expected future cash flows are comparable,
- some relevant measure of risk, when applied to both of them, gives the same value.

When the foregoing qualifications are carried on to their full swing, financial economists will regard such assets as equivalent following the law of one price. A trading-off between expected return and some risk measure underpins the substitution issue.

Remarks

Financial assets values are provided by valuation models where discounted expected cash flow models are currently used to good purpose. That risks of both transactions can be measured on an ex~ante basis, it comes as another assumption of this definition.

We can also perform arbitrage successfully by selling at an expected moment and purchasing later. If we have the asset, that means opening a short position. If we do not, we open a short-selling position, a procedure seldom used in financial markets since there are many restrictions to the activity, albeit short-selling is a crucial assumption in many Financial Economics models.

In financial environments, risk can only sparingly be dispensed with, and we say that in such cases a pure arbitrage is under way. But coping with risk is just a fact of life in capital markets. Moreover, whenever the purchase of a security at date "t_1" involves its selling at a later date "t_2", the arbitrage is risky, because at the beginning we do not know for certain the value $\mathbf{V\,(g_2;\ s_2;\ t_2\,)}$ is going to have at date "t_2" – unless we held some assets like zero-coupon bonds till maturity.

Example *A Foreign Exchange Spot-Spot Arbitrage*

An investor finds out that Argentine pesos are a little undervalued in Sao Paulo and asks his broker to firstly buy American dollars for him in Buenos Aires, to sell them later in Sao Paulo against Argentine pesos. In the framework of Definition 2:

$$g_1 = g_2 \qquad t_1 = t_2 \qquad m_1 \neq m_2$$

$$\Pi\ (\text{long-short}) = V(m_2\,;\,s) - V(m_1\,;\,l) > 0$$

No investment required; risk is kept negligible as far as the broker is trustworthy and at pains to be regarded like a reputation holder. Some interesting features, attending interest rate arbitrage, were developed by Spraos (1959).

2.1. SHORTCOMINGS IN THE STANDARD FINANCIAL ARBITRAGE MODEL : EMPIRICAL EVIDENCE

In section 1.2, it was surveyed evidence against the standard arbitrage model. Hardly surprising that intensive research has also been undertaken to uncover the weakness of an oversimplified world as that depicted in the standard financial arbitrage model. Let us highlight some contributions worthy of record and comment.

a) In practice, arbitrageurs are able of drawing up arbitrage opportunities through the exchange of substitutes which carry a legal right to acquire or dispose of, and conversely an obligation to provide or accept, securities identical with an existing security on known terms and contracts. This issued was studied by Henderson and Martine (1986), mainly through the following examples:

- Assets substitutes by risk grades, term to maturity, coupons, duration and convexity.

- Convertible bonds and New issues of ordinary shares
- Options and futures
- ADRs and US Treasury Strips
- Risk arbitrage: by opening positions in stock from companies in the threshold of mergers or acquisitions, to profit later arbitraging between them.

b) In practice, it is very difficult to carry out a complete arbitrage with financial assets, because of quantities, terms to maturity, regulations, marginal accounts balances linked with leverage and short-selling, as it was proved by Larkman (1986).

c) Bralsford (1986), working with both currencies and interest rate swaps, higlights some facts that enhance discrepancies between prices and returns, namely: mismatching of regulations, foreign exchange policies, minimal sizes required for any trade (market microstructure features)

d) Metcalf (1990) has done econometric research on departures from pure arbitrage. In the United States, there is a tax exemption granted to municipal and state bonds that has allowed municipal and state governments to profit from arbitrage opportunities in a consistent and lasting way. They issue tax-exempted bonds at a rate r(m) and invest the proceeds at taxable rate r(t). This is a clear example of what influential microstructure, transaction costs and creative accounting could be in the capital markets. Although the practice is illegal, the Internal Revenue Service has not been able to prevent state and local governments from earning arbitrage windfalls. Even the so-called Gram-Rudman-Hollings Law (balanced budget) has proved very difficult to be enforced. In fact, Metcalf singles out two kinds of arbitrage:

- *Financial Arbitrage*: the one reported above.
- *Savings Arbitrage*: raising taxes to invest in securities with high returns. The interest from investments is paid to taxpayers through lower taxes expected in the future.

e) There is an interesting case of the law of one price violation in the bonds market, provided in an empirical research by Daves y Ehrhardt (1993). Interest coupons and principal coupons are not perfect substitutes. When reconstructing them after the moment they went into the market as strips, there is a positive differential in favour of the principal coupon. By the same token, they are not equivalent as regards liquidity. Dealers rebuild the bond and sell it a higher price than the one we could get by adding the prices of the single coupons. Furthermore, the spread at the selling side is not the same as the spread at the buying side. In general, it is higher.

f) As it was argued by Shleifer and Summers (1990), arbitrage is risky and therefore limited. This approach helps to explain available anomalies in the efficient markets model, and also broad features such as trading volumes and actual investment strategies. They single out two types of economic agents in the capital markets, arbitrageurs (smart money, rational speculators) and noise traders (liquidity traders, irrational agents). The former are proficient in building-up portfolios on fully rational expectations about financial assets returns. The latter are subject to systematic biases, acting on psychological impulse (noise). It is for the arbitrageurs to bring prices towards fundamentals. Two types of risk limit arbitrage in real markets:

- *Fundamental risk:* selling short to repurchase later at a lower price may fail whenever expectations on that asset improve; that is to say, prices may never revert to fundamental values eventually.

- *Unpredictability of the future resale price*: at the time of liquidating his position in future the arbitrageur would bear the risk of more overpricing for the asset. This type of risk is financial by nature because, even if prices converge to fundamentals, the path could be long and bumpy.

Although risk makes arbitrage ineffective, pervasive limits in actual arbitrage are also to be found in the imperfect knowledge about fundamental values, and even in the ability to detect price changes that reflect deviations from fundamentals. That is to say, there are problems with mispricing identifications as well as with the risk of betting against them. News alone do not fully explain price changes, because uninformed changes in demand also have a word along the process. Finally, capital requirements for a successful arbitrage mean that money cannot be indefinitely raised, neither costlessly nor unfettered by regulations.

g) When both investors and securities are subject to differential taxation, there may be no set of prices that rule out tax arbitrage (Dammon and Green,1987). The existence of "no-tax-arbitrage" prices ensures the existence of equilibrium prices. Identical securities that contribute to taxable income to different degrees will, in general, be valued differently and equilibrium will fail to exist unless short-sale restrictions are imposed that prevent investors from exploiting such arbitrage opportunities.

h) Mitchell and Pulvino (2000) followed a sample of 4,750 stock swap mergers, cash mergers and cash tender offers during 1963 – 1998 and characterized the risk and return in risk arbitrage, finding out excess returns of 4 % per year. The risk arbitrage came out from the difference between the target's stock price and the offer price, that is to say, an arbitrage spread. The findings suggest that financial markets exhibit systematic inefficiency in the pricing of firms involved in mergers and acquisitions.

i) Arbitrage is very active between spot and future markets, with offsetting positions so that the law of one price holds. This is the usual statement, but in real world, mispricings are not infrequent and market frictions prevent arbitrage from taking place. Kempf (2001) studied short selling restrictions and early unwinding opportunities finding a major impact on the mispricing behaviour, mainly on index arbitrage trading by using the german stock index DAX and the DAX futures. The paper showed

- that arbitrageurs faced trading and holding costs, and market impact costs as well;
- the pervasive influence of asymmetric holding costs and early unwinding.

j) Mitchell, Pulvino and Stafford (2001) studied the impediments to arbitrage in equity markets between 1985 and 2000, when the market value of a company is less than the sum of its publicly traded parts. There were arbitrage opportunities, they persisted, and prices did not converge to fundamental values. In 82 situations they found out that cross holdings had brought about this environment in which main transaction costs were coming out of imperfect information and actual trades.

k) The late 1990s wave of mergers and acquisitions was intended to consolidate industries. It can be explained as a response to market misvaluation of potential acquirers, potential targets and their combinations (Shleifer and Vishny, 2001). This approach sees managers as completely rational, conversant with market inefficiencies and arbitraging them to their profit.

l) The activity of arbitrageurs impact prices, and they back their trading with their own capital (capital adequacies for trading books of banks and securities firms, the margin levels imposed by clearing brokers, margin on futures and on leveraged equity accounts). On this issue, Attari and Mello (2001), conclude that "persistent price deviations occur and then they are the result of arbitrageurs being financially constrained."

m) If arbitrage is costly and noise traders are active, assets prices will be at variance of fundamental values, as Gemmill and Thomas (2002) proved in connection with a huge number of closed-end funds. They showed that those funds performed with an average discount to fundamental values in the long run, mainly because managers set high charges.

2.2. STANDARD FINANCIAL ARBITRAGE DYNAMICS

Profiting from definition 2, the payoff operator $\Pi(\,.\,)$ for financial arbitrage processes will show the following adjustment dynamics:

$$\begin{cases} \Pi(\text{long-short}) = V(g_2; m_2; t_2; s) - V(g_1; m_1; t_1; l) \rightarrow 0 \\ \Pi(\text{short-long}) = V(g_1; m_1; t_1; s) - V(g_2; m_2; t_2; l) \rightarrow 0 \end{cases}$$

This is the place to introduce a concept that will help to keep the further discussion within operational bounds. We are speaking about the arbitrage gap, which intends to measure the expected return from an arbitrage opportunity, in nominal terms. It will be needed in the next section, devoted to transaction algebras.

Definition 3

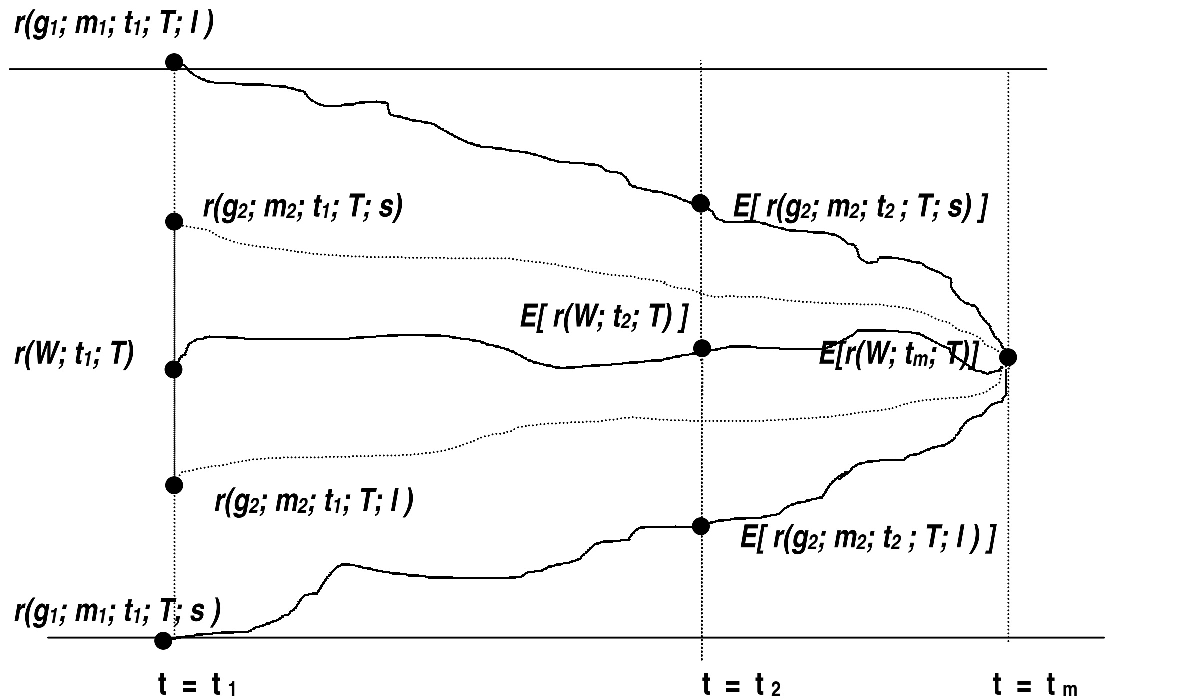
*By **Arbitrage Gap** is meant the rate of return brought about either in long-short or short-long arbitrage process, given by the following equations:*

$$\begin{cases} 1 + r(\text{arbitrage; long-short}) = V(g_2; m_2; t_2; s) \;/\; V(g_1; m_1; t_1; l) \\ 1 + r(\text{arbitrage; short-long}) = V(g_1; m_1; t_1; s) \;/\; V(g_2; m_2; t_2; l) \end{cases}$$

For any arbitrage to be successful both rates must be positive. The problem arises when transaction costs and microstructure features of the market are neglected, as it has been done by the conventional wisdom for many years, because the effective rate of return of the arbitrage cannot do without enlarged transaction costs. And this may lead to negative arbitrage gaps, preventing the economic agent from seizing the arbitrage opportunity eventually.

Whereas in section 1.3 and Exhibit 1 we worked with assets values in monetary terms, that is to say prices in markets with public offer mainly, the financial arbitrage dynamics is worthy of being developed from the point of view of rates of return. That is why Exhibit 2 seems turned upside down with respect to the former exhibit. The rationale is to be found in definition 3.

Exhibit 3: STANDARD FINANCIAL ARBITRAGE DYNAMICS



References

Horizontal axis:	*time history for the monetary valuation of the asset*
Vertical axis:	*expected returns of assets and benchmark*
E[r(W; t; T)]:	*expected return of the benchmark along the [t; T] horizon*
$t = t_1$:	*date at which an arbitrage opportunity is found out*
$t = t_2$:	*date at which the transaction is rounded off (temporal arbitrage)*
$t = t_1$:	*date at which the transaction is rounded off (spatial arbitrage)*
$t = t_m$:	*date at which the price adjustment comes to an end*
[t ; T]:	*investment horizon (T could also mean maturity date, in that case* $t_m \leq T$*)*

3. TRANSACTIONAL ALGEBRAS A NON-STANDARD APPROACH TO ARBITRAGE

Either the standard trading or the financial arbitrage model, they both abstract from market microstructure concerns. That is to say, nothing less than institutional arrangements, intermediaries and transaction costs are left out. A non-standard approach to arbitrage has to redress this tight constraint, by making explicit the world where transactions take place eventually.

Such is the approach intended in this paper. To begin with, however, we firstly must do some groundwork, namely to introduce what is meant by differential rates of return, residual information sets. In this way, the notion of transactional algebra will become meaningful and functional to understand financial arbitrages from the investor´s side.

3.1. INFORMATION SETS

The scaled change of securities prices in a period becomes its total rate of return by means of the relation

$$\mathbf{r(t,T) = [\,P(T) + I(t,T) - P(t)\,] \;/\; P(t)}$$

where P(t) and P(T) stand for prices at the beginning and the end of the holding period, whereas I(t,T) stands for cash inflows provided by the security along the holding period (like dividends or interest). Most of the time, we don't know the value of either **P(T)** or **I(t, T)** at date " **t** ". On an ex~ante basis we can substitute estimated values for **P(T)** and **I(t , T)**, namely :

$$E[\,P(T)\,] + E[\,I(t,T)\,]$$

This last value is conditional on the economic agent's "information set" Ω_t . ***An information set*** means the set of all available information to him, up to the valuation date.

Remarks

It has been customary to translate the phrase " all available information " within the realm of Fama's influential approach to efficiency in capital markets, by pointing out to weak, semi-strong or strong types of information sets (Fama 1970, 1991). An updated rendering is Elton-Gruber (1995). For a contesting approach, Shleifer's book (1999) and Daniel-Titman's paper (2000) are of interest.

From the begininig of the 1970s, transaction costs theory (Williamson, 1996) and the incomplete contracts approach (Hart, 1995) have been stressing the role bounded rationality and opportunistic behaviour perform at impairing the quality of information sets. An attempt to make operational this effort can be found in Apreda (2000a, 2000b).

Therefore, rates of change carry on a conditional feature upon future states of the world.

$$< E[\,P(T), \Omega_t\,] + E[\,I(t, T, \Omega_t)\,] > \;/\; P(t) = 1 + E[\,r(t, T, \Omega_t)\,]$$

Thus, the dated information set comes down to an actual marker for both ex~ante and ex~post assessments. On the other hand, the different measures of both ex~ante and ex~post assessments of financial assets rates of return can be attempted because we take stock on different information sets

$$\Omega_t \quad , \quad \Omega_T$$

Contrasting both information is essential, because most of the time they are different. It is for information surprises to close the gap between them. This issue has lately raised due concern among scholars and practitioners. With professor Elton's own words:

" *The use of average realized returns as a proxy for expected returns relies on a belief that information surprises tend to cancel out over the period of a study and realized returns are therefore* *an unbiased estimate of expected returns. However, I believe that there is ample evidence that this belief is misplaced"* (Elton, 1999).

Therefore, we need to set up foundations in the family of information sets which allow us to properly define not only differential rates but the expectations operator as well. To accomplish such a task we have to resort to finite algebras of sets.

Remark

In this framework of discrete modelling with finite algebras of sets, it does not seem essential to make the underlying probability distributions explicit, at least for the environment we are proposing in this paper. Whereas infinite set structures and continuous modelling have been widely undertaken, with strong stress in probability theory and the so-called sigma-algebras, the discrete approach used in this paper has been rather neglected so far.

Definition 4

A *is called an* ***algebra*** *of sets if it satisfies the following properties:*

$$E \in \boldsymbol{A} \ , \quad F \in \boldsymbol{A} \quad \Rightarrow \quad E \cup F \in \boldsymbol{A}$$

$$E \in \boldsymbol{A} \ , \quad F \in \boldsymbol{A} \quad \Rightarrow \quad E^{C} \in \boldsymbol{A}$$

That is to say, an algebra of sets is an structure closed under unions and complements of sets. Some authors, like Aliprantis (1999) or Halmos (1974), give a stylish translation of an algebra of sets as being any ring containing the space **X**. (A ring being a weaker structure, closed only for complements and unions of sets)

3.2. DIFFERENTIAL RATES OF RETURN

If we were able to access information subsets of Ω_t , this would allow us to finding out different sources which could explain the rate of change $\mathbf{r(t, T, \Omega_t)}$ as defined on the basic information set Ω_t . What, for instance, if we knew that there is a subset $\Omega^1{}_t$ of Ω_t, namely,

$$\Omega^1{}_t \subseteq \Omega_t$$

which is so influential that the rate of change $\mathbf{s^1(t, T, \Omega^1{}_t)}$ could explain two thirds of the value of $\mathbf{r(\,.\,)}$ at least?

It is within this setting that it becomes advisable to split down the original rate of return into two components:

$$\mathbf{1 + r(t, T, \Omega_t) = [\,1 + s^1(t, T, \Omega^1{}_t)\,] \,.\, [\,1 + g^1(.)\,]}$$

where

$$\mathbf{s^1(t, T, \Omega^1{}_t)}$$

stands for the rate of change conditional to the information set $\Omega^1{}_t$. Furthermore, $\mathbf{g^1(\,.\,)}$ comes up by solving the equation and stands for whatever remains of $\mathbf{r(\,.\,)}$ after taking into account $\mathbf{s^1(\,.\,)}$. So, this complementary rate $\mathbf{g^1(\,.\,)}$ fills the gap between both rates and it is conditional upon the information set

$$\Omega_t - \Omega^1{}_t$$

That means:

$$\mathbf{g^1(.) = g^1(t, T, \Omega_t - \Omega^1{}_t)}$$

Hence, we can now restate the former relationship between these three rates of change:

$$\mathbf{1 + r(t, T, \Omega_t) = [\,1 + s^1(t, T, \Omega^1{}_t)\,] \,.\, [\,1 + g^1(t, T, \Omega_t - \Omega^1{}_t)\,]}$$

The discussion above has paved the way to the following definition that accounts for differential rates and residual information sets. Although this definition keeps a very simple format lying on a general structure of sets, it can be expanded on to more complex backgrounds (details in Apreda, 2001).

Definition 5

Let ***A*** *be an algebra of sets in* ***X****. Given the rates of return* $\boldsymbol{r(t, T, \Omega_t)}$ *and* $\boldsymbol{s^1(t, T, \Omega^1{}_t)}$ *such that* $\Omega^1{}_t \subseteq \Omega_t$, *it is said that* $\boldsymbol{g^1(.)}$ *is* ***a simple differential rate*** *to both* $\boldsymbol{r(.)}$ *and* $\boldsymbol{s^1(.)}$ *if and only if*

$$\boldsymbol{g^1(.) = g^1(t, T, \Omega_t - \Omega^1{}_t)}$$

and it fulfils:

$$\boldsymbol{1 + r(t, T, \Omega_t) = [\,1 + s^1(t, T, \Omega^1{}_t)\,] \,.\, [\,1 + g^1(t, T, \Omega_t - \Omega^1{}_t)\,]}$$

Furthermore, the set $\Omega_t - \Omega^1{}_t$ *will be called* ***a residual information set****.*

Remarks

We see that $\mathbf{g^1(\,.\,)}$ is the remainder of the leading rate $\mathbf{r(\,.\,)}$, given $\mathbf{s^1(\,.\,)}$.

When plugging numbers in definition 5, it may hold true that

$$\mathbf{[\,1 + s^1(t, T, \Omega^1{}_t)\,] . [\,1 + g^1(t, T, \Omega^g{}_t)\,] = [\,1 + s^1(t, T, \Omega^1{}_t)\,] . [\,1 + h^1(t, T, \Omega^h{}_t)\,]}$$

which implies

$$\mathbf{g^1(t, T, \Omega^g{}_t) = h^1(t, T, \Omega^h{}_t)}$$

but this fact doesn't convey that, as functions, $g^1(\,.\,) = h^1(\,.\,)$, because they can be defined on different information sets.

What if we now tried to deal with $g^1(\,.\,)$ the same way we did with $r(\,.\,)$, by resorting to the simple differential rates definition? That is to say, what if knew that we can pick out another rate $s^2(\,.\,)$ conditional upon a distinctive information set $\Omega^2{}_t$ that fulfills:

$$\Omega^2{}_t \subseteq \Omega_t \quad , \quad \Omega^2{}_t \cap \Omega^1{}_t = \varnothing$$

In that case we should ask for the remainder of $g^1(\,.\,)$, and try to solve:

$$1 + g^1(t, T, \Omega_t - \Omega^1{}_t) = [\,1 + s^2(t, T, \Omega^2{}_t)\,]\,.\,[\,1 + g^2(.)\,]$$

Following this way, we may obtain a finite vector of rates of change

$$< s^1(.), s^2(.), \ldots\ldots, s^N(.) >$$

all of them stemming from the primary rate $r(\,.\,)$, and also a differential rate $g^N(\,.\,)$ performing as a remainder, so as the following relationship must hold true by iteration:

$$1 + r(t, T, \Omega_t) = \{\, \Pi_{1 \le k \le N} [\,1 + s^k(t, T, \Omega^k{}_t)\,]\,\}\,.\,[\,1 + g^N(\,.\,)\,]$$

Remark

Now, a problem certainly arises with this expression, because we have still said nothing on the underlying information sets of the components in the following finite vector

$$< g^1(.), g^2(.), \ldots\ldots, g^N(.) >$$

To handle this difficulty, we need to enlarge upon Definition 5. Such a task is accomplished in Apreda (2001a).

3.3. LOOKING FOR CONTEXTUAL SETS AND CHOOSING THE RELEVANT ALGEBRA OF INFORMATION SETS

In practice, we choose a primary information set Ω_t in the space **X**, and we are concerned with a certain family of contextual sets in **X**, which might amount to economic variables, or transaction costs structures, for instance. It is the purpose of this section to show how a family of relevant sets may be spanned into a suitable minimal algebra. An example will shed light about contextual sets.

While dealing with financial assets rates of return, we should be intent on making the analysis inclusive of the transaction costs structure. At least, five contextual subsets of **X** seems to be useful (this approach has been developed in Apreda, 2000a and 2000b) :

- Intermediation costs : INT
- Microstructure costs : MICR

- Financial costs on transactions : FIN
- Information costs : INF
- Taxes : TAX

We define a family α of contextual sets to Ω_t :

$$\alpha = \{ E , F , G , H , J \}, \quad \text{where}$$

$$E = \Omega_t \cap INT ; F = \Omega_t \cap MICR ; G = \Omega_t \cap FIN ; H = \Omega_t \cap INF ; J = \Omega_t \cap TAX$$

The minimal algebra which is spanned by these sets and Ω_t. will meet our purposes. We denote this algebra as $\mathbf{A}[\Omega_t , \alpha]$. That is to say:

$$\mathbf{A}[\Omega_t , \alpha] = \mathbf{A}[E, F, G, H, J, \Omega_t]$$

If we wanted to go beyond transaction costs, the class would be embedded into a larger one, with more factors of analysis; for instance, inflation or the rate of exchange. Making for a definition, we get:

Definition 6

Given $\Omega_t \subseteq X$, and a finite family of contextual sets to Ω_t *in X,*

$$\alpha = \{ E_1 , E_2 , E_3 , ... , E_N \}$$

the algebra $\mathbf{A}[\Omega_t , \alpha]$ spanned by the family

$$\{ E_1 \cap \Omega_t , E_2 \cap \Omega_t , E_3 \cap \Omega_t , ... , E_N \cap \Omega_t , \Omega_t \}$$

will be called the ***relevant algebra to the information set Ω_t, subject to the contextual famiy α.***

3.4. WHAT IS MEANT BY A TRANSACTIONAL ALGEBRA

We are going to introduce, therefore, the concept of a transactional algebra that intends to supply with a framework of analysis of financial trading, including arbitrage processes, embedding within it the institutional context, a convenient transaction costs function, and residual information sets. To our knowledge, this is the first time this notion is used in Financial Economics, and it was developed by the author while spending a quarter as Visiting Fellow at the Salomon Center of the Stern School of Business, New York University, in 2001. (Foundations of this approach are to be found in Apreda (2001, Working Paper Series, S- 01-3, Stern, NYU, from which we have drawn some analytical devices that were used in the foregoing groundwork)

Definition 7

*We are going to call **Transactional Algebra** a complex structure whose distinctive features are:*

> *i. Existence of one or more markets where financial assets can be exchanged through the channels of public or private placements.*
>
> *ii. The institutional framework and the market microstructure are set up by means of*
>
> > *trading and regulatory institutions;*
> > *intermediaries, investors, and regulators;*
> > *enforceable laws and rules of the game;*
> > *contractual arrangements about the property rights attached to each transaction.*
>
> *iii. A total transaction costs function is explicitly given which includes all computable enlarged transaction costs.*
>
> *iv. Residual information sets and differential rates of return are managed out of a relevant algebra of contextual sets*

3.5. ENLARGED TRANSACTION COSTS: THE COSTS OF RUNNING TRANSACTIONAL ALGEBRAS

Transaction costs are usually neglected on the grounds of being small. Furthermore, as it is stated in some quarters, some transaction costs are becoming negligible as communication devices improve. By all means, this belief is misplaced because transaction costs are the costs of running nothing less than a transactional algebra.

Firstly, what it is customarily meant by transaction costs points only at some particular types of trading costs, mainly linked with purchasing and selling securities. Although in some markets trading costs are being curbed, in other places they are not. The sensible question to elicit is about the structure of those trading costs, which is not so simple as it seems at first sight.

Secondly, enlarged transaction costs encompass a broad variety of items:

- intermediation (INT),
- microstructure (MICR),
- information (INF),
- taxes (TAX),
- and financial costs (FIN).

Although these five categories are neither exhaustive nor the only ones to work with, we believe that they allow for a sensible assessment of the transaction costs rate as

$$\mathbf{TC}\,(t_1\,;t_2\,;m_1\,;m_2\,;\ \Omega^{TC}{}_{t1})$$

which is a construct with the following features:

a) it comes by the side of every single transaction;
b) it amounts to a rate of change that may be expressed in percentage;
c) and it may be framed by means of the functional relationship of a multiplicative model:

$$< 1 + TC\,(t, T, \Omega^{TC}{}_t\,) > \; = \; < 1 + INT\,(t\,,T, \Omega^{INT}{}_t\,) > . < 1 + MICR(t, T, \Omega^{MICR}{}_t\,) > .$$

$$. < 1 + TAX\,(t, T, \Omega^{TAX}{}_t\,) > . < 1 + INF\,(t, T, \Omega^{INF}{}_t\,) > . < 1 + FIN\,(t, T, \Omega^{FIN}{}_t\,) >$$

with the restriction

$$\Omega^{k}{}_t \subseteq \Omega^{TC}{}_t \quad \text{for } k: \text{ INT, MICR, INF, FIN, TAX}$$

where $\Omega^{k}{}_t$ stands for any distinctive subset of the underlying information set to the transaction costs rate.

We are interested here in applying this functional relationship to arbitrage processes, distinguishing long from short positions, as shown below.

$$\left\{\begin{array}{ll} \textbf{short position:} & 1 + TC\,(t_1\,;t_2\,;m_1\,;m_2\,;s\,) = \\ \multicolumn{2}{l}{[\,1 - int(s)\,] . [\,1 - micr(s)\,] . [\,1 - tax(s)\,] . [\,1 - inf(s)\,] . [\,1 - fin(s)\,]} \\ & \\ \textbf{long position:} & 1 + TC\,(t_1\,;t_2\,;m_1\,;m_2\,;l\,) = \\ \multicolumn{2}{l}{[\,1 + int(s)\,] . [\,1 + micr(s)\,] . [\,1 + tax(s)\,] . [\,1 + inf(s)\,] . [\,1 + fin(s)\,]} \end{array}\right. \tag{8}$$

When selling, costs lessen the cash flows to be finally collected. When purchasing, they add to incurring outflows.

Remark

Each component has its own functional structure which does not come up as linear, necessarily. In fact, non-linearity is customary and useful in standard research, which take advantage of piece-wise linear functions, or still better, the so called simple or step functions, so as to approximate more complex relationships. [For instance, Levy-Livingston (1995) on portfolio management, Day (1997) in nonlinear dynamics applied to economics, are good sources]

Before concluding this section, we need to take a step further and embed each transaction cost rate pertaining the short and long position into a comprehensive differential rate that account for the costs of running transactional algebras.

Definition 8

In a transactional algebra setting, let us denote with

$$\textit{diff TC}$$

the differential transaction costs rate that solves the equation

$$1 + \textit{diff TC} = [\,1 + TC\,(\,t_1\,;t_2\,;m_1\,;m_2\,;s\,)\,] \,/\, [\,1 + TC\,(\,t_1\,;t_2\,;m_1\,;m_2\,;l\,)\,]$$

In other words, the differential transaction costs rate measures up the whole impact of the costs of running the transactional algebra involved with an arbitrage. It will prove functional for dealing with the main statements of Lemmas 1 and 2 that will be proved in next section, whereas its practical implications will become clear in sections 4.1 and 4.2.

4. ARBITRAGE WITHIN TRANSACTIONAL ALGEBRAS

Rates of return must be broken down into the cost components on the one hand, and a return netted from them, on the other hand (foundations for this statement is to be found in chapter 6). To deal with this issue two lemmas follow.

Lemma 1: In a transactional algebra, and for every arbitrage opportunity:

i) there is an arbitrage return net of transaction costs.

ii) there are diferential rates to translate each type of transaction costs arising from the rounding-off trades;

Proof:

i. Using (8), the money to collect when selling the asset (all-in-cost basis) would amount to

$$V(\,s\,)\,.\,[\,1 - int(s)\,]\,.\,[\,1 - micr(s)\,]\,.\,[\,1 - tax(s)\,]\,.\,[\,1 - inf(s)\,]\,.\,[\,1 - fin(s)\,] =$$

$$= V(\,s\,)\,.\,[\,1 + TC\,(\,t_1\,;t_2\,;m_1\,;m_2\,;s\,)\,]$$

and when purchasing the asset (all-in-cost basis):

$$V(l).[1 + int(l)].[1 + micr(l)].[1 + tax(l)].[1 + inf(l)].[1 + fin(l)] =$$

$$= V(l).[1 + TC(t_1;t_2;m_1;m_2;l)]$$

The return from this arbitrage yields (either trading or financial, and in nominal terms) follows from definition 3

$$V(s) / V(l) = 1 + r(arbitrage)$$

When taking transaction costs arising from the rounding-off trades we can put forth the equation

$$V(s).[1 + TC(t_1;t_2;m_1;m_2;s)] / V(l).[1 + TC(t_1;t_2;m_1;m_2;l)] = \tag{9}$$

$$= 1 + r_{net}(arbitrage)$$

and solving for r_{net} *(arbitrage)* we have a measure of the return of the arbitrage that embodies transaction costs.

By resorting to definition 5, the differential transaction costs rate follows from

$$1 + diff\,TC = [1 + TC(t_1;t_2;m_1;m_2;s)] / [1 + TC(t_1;t_2;m_1;m_2;l)]$$

the relationship (9) can now be rewritten in a much more compact format:

$$[V(s) / V(l)].(1 + diff\,TC) = 1 + r_{net}(arbitrage)$$

and, last of all, we get

$$[1 + r(arbitrage)].(1 + diff\,TC) = 1 + r_{net}(arbitrage)$$

ii) Let us substitute now the labels t_k (k: 1, 2, 3, 4, 5) for the enlarged transaction costs labels trad, micr, tax, inf, and fin, respectively. Then,

$$1 + g_k = [1 - t_k(s)] / [1 + t_k(l)]$$

where g_k performs as a differential rate drawn out of $t_k(s)$ and $t_k(l)$.

On the other hand, we can replace in (8) to get the equivalence:

$$1 + diff\,TC = [\Pi(1 + g_k)]$$

$$= [1 + TC(t_1;t_2;m_1;m_2;s)] / [1 + TC(t_1;t_2;m_1;m_2;l)]$$ **END.**

Although in the standard financial arbitrage model arbitrage opportunities can be grabbed once the conditions of definition 2 are met, in a transactional algebra structure this cannot be granted. In fact, arbitrage will only be feasible whenever the arbitrage gap overreaches the constraints of the transactional algebra, as the following lemma makes clear.

Lemma 2: In a transactional algebra, the fulfillment of the standard financial arbitrage conditions, it does not grant that an arbitrage opportunity remains profitable.

Proof:

Whenever an investor takes advantage of arbitrage opportunities, he tries to lock in a sure profit that follows from definition 3:

$$1 + r(\text{arbitrage; long-short}) = V(g_2; m_2; t_2; s) \; / \; V(g_1; m_1; t_1; l)$$

$$1 + r(\text{arbitrage; short-long}) = V(g_1; m_1; t_1; s) \; / \; V(g_1; m_2; t_2; l)$$

Let us analyze each relationship at a turn.

a) <u>*the long-short type of arbitrage*</u>

If we included transaction costs, according with Lemma 1, we would get a net arbitrage return:

$$1 + r_{net}(\text{arbitrage}) = [1 + r(\text{arbitrage})] \, . \, [1 + \text{diff TC}]$$

For the arbitrage to become successful, it must hold:

$$[1 + r(\text{arbitrage; long-short})] = [V(g_2; m_2; t_2; s) \; / \; V(g_1; m_1; t_1; l)] > 1$$

but this is not a sufficient feature, because differential transaction costs could lead to

$$1 + r_{net}(\text{arbitrage; long-short}) = [1 + r(\text{arbitrage}); \text{long-short}] \, . \, [1 + \text{diff TC}]) < 1$$

thus yielding a negative return on the net rate of return for the arbitrage.

b) <u>*the short-long type of arbitrage*</u>

By the same procedure as in a) we would get that is not enough the fulfillment of

$$[V(g_1; m_1; t_1; s) \; / \; V(g_2; m_2; t_2; l)] > 1$$

because the transaction costs structure could bring about the following outcome:

$$1 + r_{net}(\text{arbitrage; short-long}) = [1 + r(\text{arbitrage}); \text{short-long}] \, . \, [1 + \text{diff TC}] < 1$$

giving forth a negative return on the net rate of return for the arbitrage. <u>***END.***</u>

It is from Lemma 2 that we can frame a definition of what is meant by financial arbitrage within a transactional algebra.

Definition 9

Financial Arbitrage within a Transactional Algebra *is a decision making process whose main features are:*

i. *the trade of a financial asset "g_1", at an expected moment "t_1", in a certain market "m_1", at the value* $\boldsymbol{V(g_1; m_1; t_1)}$ *;*

ii. *the trade of a financial asset "g_2", at an expected moment "t_2", in a certain market "m_2", at the value* $\boldsymbol{V(g_2; m_2; t_2)}$, with $\boldsymbol{t_1 \leq t_2}$ *;*

iii. *making a sure profit from round-off transactions, either the long-short or the short-long types, that is to say, the payoff functions* $\Pi(\,.\,)$ *are positive:*

$$\begin{cases} \Pi(\text{long-short}) = 1 + r(\text{arbitrage; long-short}) = V(g_2; m_2; t_2; s) \;/\; V(g_1; m_1; t_1; l) > 1 \\ \Pi(\text{short-long}) = 1 + r(\text{arbitrage; short-long}) = V(g_1; m_1; t_1; s) \;/\; V(g_1; m_2; t_2; l) > 1 \end{cases}$$

iv. *no investment is required for rounding off both transactions;*

v. *risks of rounding off both transactions are null.*

vi. *it meets the following boundary conditions*

$$1 + r_{net}(\text{arbitrage; long-short}) = [\,1 + r(\text{arbitrage}); \text{long-short}\,] \,.\, [\,1 + \text{diff TC}\,]) > 1$$

$$1 + r_{net}(\text{arbitrage; short-long}) = [\,1 + r(\text{arbitrage}); \text{short-long}\,] \,.\, [\,1 + \text{diff TC}\,] > 1$$

4.1. PRACTICAL CONSEQUENCES OF LEMMA 1 AND LEMMA 2

We are going to point out to a pair of consequences that are worthy of further comment.

a) Lemma 1 brings about a strong outcome:

$$1 + r_{net}(\text{arbitrage}) = [\,1 + r(\text{arbitrage})\,] \,.\, [\,1 + \text{diff TC}\,]$$

that is to say, in order to make a profitable arbitrage within a transactional algebra we should check out whether the nominal gross gap promised by the arbitrage opportunity is wide enough to cover the differential transaction costs rate. And Lemma 2 shows that decision-making follows only when this check is carried out eventually.

On the other hand, it is expected most of the time that

$$\mathbf{1 + diff\ TC < 1}$$

because the final blend of long and short positions (regarded from the transaction costs involved with them), it makes the differential rate negative, meaning that the final action of the transactional algebra amounts to lessening the gross arbitrage gap. But this boundary condition does not prevent some particular situations in which the differential rate become positive (in this case the costs of the long position are negligible or becomes negative); if such were the environment, the arbitrage rate would be reinforced, by all means.

b) From Lemma 1 we know that

$$1 + r_{net}\ (arbitrage) = [\ 1 + r(arbitrage)\ ]\ .\ [\ 1 + diff\ TC\ ]$$

that allows for an equivalent translation by applying the concept of reverse differential rate to ***diff TC***,

$$[\ 1 + diff\ TC\ ]\ .\ [\ 1 + rev\ diff\ TC\ ] = 1$$

and solving for the reverse of ***diff TC***

$$[\ 1 + rev\ diff\ TC\ ] = [\ 1 + diff\ TC\ ]^{-1}$$

we can have a symmetric copy of the main outcome conveyed by Lemma 1:

$$[\ 1 + r(arbitrage)\ ] = [\ 1 + r_{net}\ (arbitrage)\ ]\ .\ [\ 1 + rev\ diff\ TC\ ]$$

This outcome can be used as follows: in order to have a profitable arbitrage, the joint action of the differential transaction costs and the minimal gap that makes the arbitrage worthy of being carried out, must be at least as big as the expected gross arbitrage gap.

4.2. NUMERICAL APPLICATION TO AN ARBITRAGE IN A TRANSACTIONAL ALGEBRA

To profit from lemma 1 and lemma 2, nothing better than working out a numerical illustration so as grasp the substance of both statements.

Setting: Let us suppose that an analyst finds out that certain financial asset A shows an expected rate of return in the market

$$E[\ r(\ A\ ),\ market] = 0.0800$$

that overcomes the expected rate of return assessed from a good evaluation model widely used by the industry

$$E[\ r(\ A\ ),\ model] = 0.0750$$

Working out an arbitrage plan

Firstly, the analyst proceeds to measure the gross arbitrage gap by means of the following differential rates equation:

$$1 + E[r(A), market] = (1 + E[r(A), model]) . (1 + r(arbitrage))$$

and, solving for the arbitrage gap, he gets

$$(1 + r(arbitrage)) = 1.0800 / 1.0750$$

$$(1 + r(arbitrage)) = 1.0047$$

and finally

$$r(arbitrage) = 0.0047$$

that is to say, the analyst bets there would be 47 basis points to be grabbed from the arbitrage opportunity.

Secondly, he seeks for a close substitute of asset A in the market, let us call it B, that meets the constraint

$$E[r(B), market] = E[r(A), model]$$

For the sake of the application, we assume that the analyst successfully matches A with B.

Last of all, the functional relationship to take into account becomes:

$$1 + E[r(A), market] = (1 + E[r(B), market]) . (1 + r(arbitrage))$$

and by using lemma 1,

$$1 + r_{net} (arbitrage) = [1 + r(arbitrage)] . [1 + diff\ TC]$$

the analyst can start to look for the feasibility of long-short type of arbitrage where he opens.

- a long position in A
- a short position in B

Feasibility of the arbitrage

As he has assessed the gross arbitrage ***r(arbitrage)***, he now needs some data that he request to his fellows at the back office (see table below).

transaction costs % / position	intermediation costs	microstructure costs	taxes	information costs	Financial costs
Long A	.5	.3	.1	.1	.4
Short B	.4	.3	.1	.2	.3

By taking advantage of the relationships established in (8)

$$\begin{cases} \textbf{short position:} \quad 1 + TC(t_1 ; t_2 ; m_1 ; m_2 ; s) = \\ [\,1 - int(s)\,] \,.\, [\,1 - micr(s)\,] \,.\, [\,1 - tax(s)\,] \,.\, [\,1 - inf(s)\,] \,.\, [\,1 - fin(s)\,] \\ \textbf{long position:} \quad 1 + TC(t_1 ; t_2 ; m_1 ; m_2 ; l) = \\ [\,1 + int(s)\,] \,.\, [\,1 + micr(s)\,] \,.\, [\,1 + tax(s)\,] \,.\, [\,1 + inf(s)\,] \,.\, [\,1 + fin(s)\,] \end{cases}$$

he can thus figure out both legs of the transaction cost function straigthforwardly:

firstly, for the short position

$$1 + TC(t_1 ; t_2 ; m_1 ; m_2 ; s) =$$

$$= [\,1 - int(s)\,] \,.\, [\,1 - micr(s)\,] \,.\, [\,1 - tax(s)\,] \,.\, [\,1 - inf(s)\,] \,.\, [\,1 - fin(s)\,]$$

$$= [\,1 - .004\,] \,.\, [\,1 - .003\,] \,.\, [\,1 - .001\,] \,.\, [\,1 - .002\,] \,.\, [\,1 - .003\,] = 0.9871$$

and secondly, for the long position

$$1 + TC(t_1 ; t_2 ; m_1 ; m_2 ; l) =$$

$$= [\,1 + int(s)\,] \,.\, [\,1 + micr(s)\,] \,.\, [\,1 + tax(s)\,] \,.\, [\,1 + inf(s)\,] \,.\, [\,1 + fin(s)\,]$$

$$= [\,1 + .005\,] \,.\, [\,1 + .003\,] \,.\, [\,1 + .001\,] \,.\, [\,1 + .001\,] \,.\, [\,1 + .004\,] = 1.0141$$

Now, he is ready to figure out the differential rate associated to the transaction algebra:

$$1 + diff\ TC = [\,1 + TC(t_1 ; t_2 ; m_1 ; m_2 ; s)\,] \,/\, [\,1 + TC(t_1 ; t_2 ; m_1 ; m_2 ; l)\,]$$

$$1 + diff\ TC = 0.9871 \,/\, 1.0141 = 0.9734$$

Last of all, he uses lemma 1:

$$\mathbf{1 \; + \; r_{net} \; (arbitrage) \;\; = \;\; [\, 1 + \; r(arbitrage) \,] \;\; . \; [\, 1 \; + \; diff \; TC \,]}$$

to assess the net arbitrage gap:

$$\mathbf{1 \; + \; r_{net} \; (arbitrage) \;\; = \;\; 1.\,0047 \;\; . \; 0.9734 \;\; = \; 0.9780}$$

and by lemma 2 he has to reject this arbitrage design. In fact,

$$\mathbf{r_{net} \; (arbitrage) \;\; = - \, 0.0220}$$

If we had regarded the net arbitrage gap from the viewpoint of ***rev diff TC***, the argument would have lead to

$$\mathbf{[\, 1 + \; r(arbitrage) \,] \;\; = \;\; [\, 1 \; + \; r_{net} \; (arbitrage) \,] \, . \, [\, 1 \; + \; rev \; diff \; TC]}$$

and plugging the numbers, we would have got

$$\mathbf{1 \; + \; rev \; diff \; TC \;\; = \; 1.0273}$$

that adds up to 226 basis point over the 47 basis points produced by the gross arbitrage gap and so turning it unfeasible. In other words, the net arbitrage gap will become negative.

CONCLUSIONS

The standard trading arbitrage model provides with a simple setting and adjustment mechanisms to take profit whenever an arbitrage opportunity comes up. However, empirical evidence has been piling up showing this point of view suffers from many downsides and there are distinctive issues that remain unexplained. On the other hand, a similar environment of shortcomings can be attached to the standard financial arbitrage model.

To go beyond the limitations of the standard model, this paper introduced the notion of a transactional algebra, which shapes the arbitrage rates of return, or arbitrage gaps, within institutional settings to give account of market microstructure features and enlarged transaction costs. The main outcomes of such approach can be summarized this way:

a) arbitrage gaps can be broken down into a net arbitrage return and a differential rate that encompasses the costs of running a transactional algebra;

b) the arbitrage becomes feasible only when overcomes the differential rate of transaction costs due to the transaction algebra.

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