Order-restricted preferences and strategy-proof social choice rules*

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Abstract

Preference profiles are order-restricted (Rothstein [17] [18]) if for any pair of alternatives, x and y, the set of agents I can be partitioned in three (integer) intervals, $I_1 = [0, i_1]$, $I_2 = [i_1 + 1, i_2]$ and $I_3 = [i_2+1, |I|]$, such that I_1 is the set of agents that prefer x to y, I_2 the set of agents indifferent between both alternatives, while I_3 represents the set of those agents preferring y to x. This condition has been proven to be useful in different models of collective decision-making, where there is a *natural* ordering of individuals rather than of the alternatives.

The purpose of this article is to analyze whether or not there exists nontrivial social choice rules, defined on this preference domain, which satisfy the well-known non manipulability condition called *strategy-proofness*. By means of a simple argument, we show first that, if preference profiles are order-restricted, then the median choice rule is strategy-proof. That is, we show that, for the median choice rule, no profitable deviation can occur because of the internal coordination among the individuals that the structure of preferences induces. This result, which extends the work of Moulin, Barberá and others to a different class of domain restriction, has interesting consequences for many problems of political economy. In particular, the result is used

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here to prove the implementability of Rothstein's [18] Representative Voter Theorem in dominant strategies. *JEL Classification*: C70, C72, D71, D78.

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1 Introduction

Consider a society which must choose a collective alternative or a policy outcome from a subset X of the one-dimensional Euclidean space. If no restriction is placed on the admissible preferences of agents and $|X| > 2$, then any mechanism which takes these preferences into account for reaching a decision must either be trivial or manipulable. This disappointing result, known in the literature as the Gibbard-Satterthwaite Theorem, was the starting point of a vast research agenda on the strategic aspects of collective decision-making.

Two approaches were followed to overcome the Gibbard-Satterthwaite Impossibility Theorem. The first one consists in weakening the concept of manipulation used in the analysis. The most commonly notion employed, *strategy-proofness*, is a very strong form of robustness against manipulation. A choice rule that maps preference profiles into social alternatives is called *strategy-proof* if telling the truth is a dominant strategy for every agent. This means that, no matter what the others report, a lie never pays off. Concepts such as Nash, subgame perfect or Bayesian equilibria may be shown to be both more meaningful and more useful than dominant strategy equilibrium.

Nevertheless, if we insist on strategy-proofness, a second possible way out of the Gibbard-Satterthwaite Theorem is to restrict the domain of admissible preferences. While the *universal domain* assumption assumed by the theorem may be natural when the set of alternatives has no particular structure, it could be unreasonable strong when that set arises from some specific economic or political problem.

In effect, an important class of preferences for public economics and applied political theory is the class of *single-peaked* profiles. Intuitively, a single-peaked profile is one in which the set of alternatives can be ordered along a left-right scale in such a way that each individual has a unique most preferred alternative (or ideal point) and the individual's ranking of other alternatives falls as one moves away from her ideal point. Such profiles capture the common intuition that, for example, an individual has a most preferred ideological position on some liberal-conservative spectrum and the more distant is a candidate's ideological position from this most preferred point the more the individual dislikes the candidate.

What makes the assumption of single-peaked preferences attractive is that for this type of restricted domain there exists a wide class of strategy-proof social choice rules. The simplest one, that was first studied by Black [8] and drew continued attention ever since, is the rule that chooses for each preference profile the median of the peaks of the preferences. This mechanism, sometimes called the *median (peak) choice rule*, was then extended by Moulin [14] to other biased procedures (such as the *highest peak rule*) by assuming the existence of a number of *phantom voters*. It turns out that such mechanisms, called *extended median rules*, fully characterize strategy-proof rules among those requiring agents to reveal only their preferred alternative.

As a corollary, the results above imply that the Median Voter Theorem has a non-cooperative strategic foundation, in the sense that it is possible to construct a game that implements the median voter's most preferred alternative in dominant strategies. A simple example of such a game form is a situation in which each voter's strategy is simply to choose an alternative from X, with the outcome function then being the selection of the median of the chosen alternatives. It is easy to show that in such game each player has a dominant strategy that consists in choosing her ideal point, since selecting any other alternative can only move the median away from her ideal policy.

Although single-peakedness can be seen as an attractive preference domain for analyzing strategy-proofness, its usefulness is restricted at least in the following two ways. First, assuming that individual preferences are single-peaked is not always reasonable. For example, in the standard "one public good-one private good" model of public economics, if the public good production cost schedule is strictly concave (because the technology is subject to increasing returns to scale) as opposed to convex, then the *induced* preferences need not be single-peaked.

Secondly, social choice rules that are strategy-proof in the unrestricted domain of alternatives may not be so on arbitrary *agendas* or subsets of X. Since single-peakedness does not restrict too much the direction of preferences among alternatives that are not top, unless they lie in rather specific positions, if agents were required to vote for their top on a given range, and their unconditionally best alternative were no longer available for some or all individuals, then there will be sufficiently room for manipulation.

As we said before, the problem arises from the fact that assuming single-peakedness on the set X of conceivable alternatives need not imply many restrictions on the preferences over a given subset $\tilde{X} \subset X$ of feasible alternatives. It all depends on the shape of $\tilde{X}$. It may even be that every preference on $\tilde{X}$ is the restriction of some single-peaked preference on X. Thus, further restrictions are necessary for positive results.[1]

In recent years, an alternative class of preference profiles, that deals with both criticisms, has received increasing attention. This family of preferences, first formally introduced by Rothstein [17] [18], is commonly known as *order-restricted* preferences and it is characterized not by an ordering of alternatives but rather by an ordering of individuals.

The idea behind order-restriction is that in many circumstances ordering people is more natural than ordering alternatives. This is the case, for example, of redistributive policies, where policy-makers are concerned with reallocating resources from rich to poor people subject to the constraint that such redistributions do not reverse the rank-order of individuals' wealth. Thus, while there does not exist an obvious ordering of alternative distributions of wealth, there does exist a natural ordering of individuals in terms of individual wealth.

Unlike single-peakedness, order-restriction imposes limitations on the character of voter heterogeneity rather than on the shape of individual preferences. Under order-restricted preferences, individuals are assigned a position along a left-right scale with the condition that, for any pair of alternatives, the set of individuals preferring one of the alternatives all lie to one side of those who prefer the other.

While similar in spirit to single-peakedness, it is easy to show that neither set of restricted profiles contains the other. Furthermore, both assumptions yield that the median voter's ideal point is a Condorcet winner. But there is a subtle difference in the meaning of these results. As Rothstein [18] has

[1] As Barberá et al. [6] have shown, positive results still hold if preference domains are restricted to the set of single-peaked preferences closed on $\tilde{X}$, that is, the set of all preferences whose unconditional peaks happens to lie in $\tilde{X}$, or further, but not if we enlarge them to allow for preferences with infeasible tops. However, since it is often the case that we prefer what we cannot get, these closed preference domains are very narrow indeed.

shown, order-restricted preferences imply that the median voter is also a *representative voter*.[2] This means that, for any pair of alternatives x and y, (not just for the median top), say $x < y$, if the median voter prefers x, then all voters to his left agree with him; and, if the median voter prefers y, then all voters to his right agree also with him. In other words, there exists always a majority that agrees with the median voter, so that the majority preference relation basically coincides with the preference ordering of the median.

In this paper we analyze the existence of non-trivial strategy-proof social choice rules on the domain of order-restricted preferences. We consider such analyzes relevant because order-restriction has been shown to be an important subclass of individual preferences. In effect, in a recent paper Gans and Smart [11] have proven their relationship to the general, ordinal notion of *single-crossing* proposed by Milgrom and Shannon [13], which is shown to be sufficient for a majority voting equilibrium to exist.[3] That condition is itself related to the literature of monotone comparative statics and to the more familiar Spence-Mirlees single-crossing, or sorting, condition, so frequently used in several fields such that voting theory, mechanism design, principal-agent theory and information economics.

The paper proceeds as follow. Section 2 provides notation and definitions. In Section 3, we present and prove our main results. By means of a simple argument, we show first that, if preference profiles are order-restricted, then the median choice rule is strategy-proof. That is, we show that, for the median choice rule, no profitable deviation can occur because of the internal coordination among the individuals that the structure of preferences induces. This result, which extends the work of Moulin, Barberá and others to a different class of domain restriction, has interesting conse-

[2] A *representative voter* is an individual whose strict preference for any alternative x over any alternative y implies: (1) that x strictly defeats y by majority rule, if there are an odd number of voters; and (2) that x weakly defeats y otherwise. This result holds by the median voter if x is his ideal point or if the preferences satisfy a generalized symmetry property, but not in general.

[3] For example, in Roberts' [16] model, which analyzes the collective choice of redistributive tax and transfer rates schemes, single-peakedness may fail to account for the individual preferences (with the obvious consequence that the majority preference relation may be intransitive). In that case, Roberts invokes a condition he calls *hierarchical adherence*, which implies single-crossing, to show that if it holds, there still exists a majority preference with the desired property of quasi-transitivity.

quences for many problems of political economy. In particular, the result is used here to prove the implementability of Rothstein's [18] Representative Voter Theorem in dominant strategies. Section 4 concludes.

2 Preliminaries

The basic model of order-restricted preferences assumes that the set of agents I is finite and its cardinality $|I| = n$ odd. Without loss of generality we represent I as $I = \{1, 2, \ldots, n\}$. The set of alternatives is $X = \{x_1, ..., x_k\} \subset \Re_+$, while $\tilde{X} = \{x_1, ..., x_s\}$ represents a generic subset of X. We use χ to represent the set of all nonempty subsets of X, $\chi = \{\tilde{X} | \tilde{X} \in 2^X \backslash \emptyset\}$. In words: X is the universal set of outcomes, whereas a particular situation, or agenda, involves a $\tilde{X} \in \chi$.

Call $SW(X)$ the set of all weak (complete and transitive) and strict orderings $\succ$ on X. A maximal set associated with $\langle X, \succeq \rangle$ is $M(X, \succeq) = \{x \in X | \forall y \in X, x \succeq y\}$. $M(X, \succeq)$ gives those alternatives that are top-ranked in X with respect to the preference relation $\succeq$. Note that if both x and y are in $M(X, \succeq)$, then it must be the case that $x \sim y$. But, since we assume that orderings in $SW(X)$ are strict, the corresponding maximal sets are singletons.

Given $\succ \in SW(X)$, we define $\tilde{\succ}$ as its *induced* preference ordering on $\tilde{X}$,

$$\forall x, y \in \tilde{X},\ x \tilde{\succ} y \quad \text{if and only if} \quad x \succ y$$

A profile $\langle \succ_1, \ldots, \succ_n \rangle \in [SW(X)]^n$ is *order-restricted* on X if and only if $\exists \gamma : I \rightarrow I$ such that $\forall x, y \in X$,[4]

$$\{\gamma(i) \in I | x \succ_{\gamma(i)} y\} \gg \{\gamma(i) \in I | y \succ_{\gamma(i)} x\}$$

or

$$\{\gamma(i) \in I | x \succ_{\gamma(i)} y\} \ll \{\gamma(i) \in I | y \succ_{\gamma(i)} x\} \qquad OR - 1$$

$OR(X) \subset SW(X)$ will denote the set of all order-restricted preferences on X.[5]

[4]For any two nonempty finite sets of integers K and J define the binary relation $\gg$ as $K \gg J$ if $min\{k \in K\} > max\{j \in J\}$, i.e. if the smallest element in K is greater than the greatest element in J. If both sets are empty, let $K \gg J$ and $J \gg K$.

[5]This notion can be easily extended to non-strict orderings, $\langle \succeq_1, \ldots, \succeq_n \rangle$, by requiring

Another property, *single-crossing*, has been proven to be equivalent to $OR-1$ (Gans and Smart [11]). Thereafter, we will use it instead of the original characterization,[6]

$$\forall y > x, \forall j > i, y \succ_i x \rightarrow y \succ_j x \qquad OR-2$$

Order-restriction and single-peakedness are obviously independent conditions.[7] In particular, preferences can be order-restricted but not single-peaked. In order to illustrate this, suppose a social situation with three individuals, indexed $1, 2$ and 3, and three alternatives x, y and z in $\Re_+$, such that $x < y < z$. Assume that the agents have the following orderings (weak but not necessarily strict): $x \succ_1 y \sim_1 z$; $y \sim_2 z \succ_2 x$; and $z \succ_3 y \succ_3 x$. It is easy to show that this profile satisfies order-restriction, but not single-peakedness.

Let $\tilde{X} \in \chi$ be a subset of X, with $|\tilde{X}| = 2q+1$, $q \geq 0$. The median of $\tilde{X}$, denoted by $med(\tilde{X}) = \tilde{x}_m$, is such that $|\{x_i \in \tilde{X} | x_i \leq x_m\}| \geq q+1$ and $|\{x_i \in \tilde{X} | x_m \leq x_i\}| \geq q+1$.

A social choice function f on $\langle X, SW(X)\rangle$ is a function $f : [SW(X)]^n \rightarrow X$. A social choice function $f^m : [SW(X)]^n \rightarrow X$ is called a *median choice rule* if,

$$\forall \succ \in [SW(X)]^n, f^m(\succ) = med\{t(\succ_i)\}_{i \in I}$$

where $t(\succ_i) \in M(X, \succ_i)$. Notice that if $\succ_i$ is single-peaked on X, then $t(\succ_i)$ represents agent i's ideal point on X.

A choice rule $f : [SW(X)]^n \rightarrow X$ is *tops-only* if,

$$\forall \succ, \succ' \in [SW(X)]^n,\ f(\succ) = f(\succ') \quad \text{whenever} \quad M(X, \succ_i) = M(X, \succ'_i)\ \forall i \in I$$

A social choice function f on $\langle X, SW(X)\rangle$ is *strategy-proof* iff,

$$f(\succ_i, \succ_{-i}) \succ_i f(\succ'_i, \succ_{-i}),\ \forall i \in I,\ \succ_i, \succ'_i \in SW(X) \text{ and } \succ_{-i} \in [SW(X)]^{n-1}$$

that,

$$\{\gamma(i) \in I | x \succ_{\gamma(i)} y\} \gg \{\gamma(i) \in I | y \sim_{\gamma(i)} x\} \gg \{\gamma(i) \in I | y \succ_{\gamma(i)} x\}$$

or

$$\{\gamma(i) \in I | x \succ_{\gamma(i)} y\} \ll \{\gamma(i) \in I | y \sim_{\gamma(i)} x\} \ll \{\gamma(i) \in I | y \succ_{\gamma(i)} x\}$$

[6] The definition in $OR-2$ assumes that an appropriate relabeling of the agents' indexes, $\gamma(\cdot)$, has been already applied.

[7] Formally, a preference profile $\langle \succ_1, \ldots, \succ_n \rangle \in [SW(X)]^n$ is single-peaked on X if for all $i \in I$, there exists $t(\succ_i) \in X$ such that (1) $t(\succ_i) \succ_i x$, for all $x \in X \backslash \{t(\succ_i)\}$; (2) $y < x \leq t(\succ_i)$ implies $x \succ_i y$, and (3) $t(\succ_i) \leq x < y$ implies $x \succ_i y$.

If a social choice function f is not strategy-proof, then there exist i, $\succ_i$, $\succ_i'$ and $\succ_{-i}$ such that $f(\succ_i', \succ_{-i}) \succ_i f(\succ_i, \succ_{-i})$. We then say that f is *manipulable* at $(\succ_i, \succ_{-i})$, by i, via $\succ_i'$.

3 Results

Order-restricted preferences exhibit some characteristics that will be useful in the analysis of strategy-proofness. The first one shows that, unlike single-peakedness, it holds for every subset of alternatives:

Lemma 1 *If* $\succ = \langle \succ_1, \ldots, \succ_n \rangle$ *is order-restricted on* X*, then* $\tilde{\succ} = \langle \tilde{\succ}_1, \ldots, \tilde{\succ}_n \rangle$ *is order restricted on any agenda* $\tilde{X}$*, for all* $\tilde{X} \in \chi$*.*

Proof. By way of contradiction, assume $\tilde{\succ}$ does not satisfy order-restriction on $\tilde{X}$. That is, suppose $OR-2$ does not hold on $\tilde{X}$. Then, there exist $x, y \in \tilde{X}$, and $i, j \in I$ such that $y > x$ and $j > i$, and $y \tilde{\succ}_i x$ but $\neg[y \tilde{\succ}_j x]$. Hence, by the definition of $\tilde{\succ}$, $y \succ_i x$ and $\neg[y \succ_j x]$, which contradicts our assumption that $\succ$ is order-restricted on X. 2

Notice that Lemma 1 holds even if we allow for indifference between alternatives. In that setting, single-crossing requires that for all $y > x$, $j > i$, $y \succeq_i x$ implies $y \succeq_j x$. The reader could check that a proof similar to the one given above may be established in that case.

Conversely, even if we admit a *weak* version of single-peakedness,[8] it seems that preferences that are single-peaked on the full set of alternatives need not be so on arbitrary agendas or subsets of policies. This is a significant difference between both preference domains. In particular, the invariance of order-restriction in the real line implies that, if a social choice rule $f : [OR(X)]^n \rightarrow X$ is strategy-proof, then it must be also non-manipulable in every agenda $\tilde{X} \subset X$. That is, it must be strategy-proof no matter what subsets of policies we are considering in the collective choice process.

[8]Barberá and Jackson [5] define *weakly* single-peaked preferences in the following way. For any $X \subset \Re$, a preference relation $\succeq$ is *weakly* single-peaked on X iff there exist alternatives $t_1, t_2 \in X$ (the peaks of $\succeq$ on X), with $t_1 < t_2$, such that: (a) $t_1 \sim t_2$; (b) $[t_1, t_2] \cap X = \{t_1, t_2\}$; (c) $x < y \leq t_1 \rightarrow t_1 \succeq y \succ x$; and (d) $t_2 \leq y < x \rightarrow t_2 \succeq y \succ x$.

Conditions (a) and (b) seem to be more restrictive than the others. In fact, it is possible to conceive many situations where one of them or both at the same time fail to hold.

On the other hand, the assumptions of strict, complete and transitive individual orderings imply that for each subset of alternatives the restriction of order-restricted preferences picks out a single maximal element:

Proposition 1 *For all $\tilde{X} \in \chi$ and $\tilde{\succ}_i \in SW(\tilde{X})$, agent i has a unique top alternative $\tilde{t}_i \in \tilde{X}$.*

Proof. Trivial. Consider $\tilde{X} \in \chi$ and i's restricted preferences on it, $\tilde{\succ}_i \in SW(\tilde{X})$. Since $\tilde{X}$ is finite and $\tilde{\succ}_i$ is complete and strict, $\langle \tilde{X}, \tilde{\succ}_i \rangle$ constitutes a chain. By Zorn's Lemma there exists a maximal element for $\tilde{\succ}_i$, $\tilde{t}_i \in \tilde{X}$. 2

Another property that follows immediately is that the median choice rule on the domain of order restricted preferences requires only the information provided by the maximal elements in the individual orderings:

Lemma 2 *$f^m : [OR(X)]^n \rightarrow X$ is tops-only.*

Proof. Suppose two preference profiles $\bar{\succ}, \hat{\succ} \in [OR(X)]^n$, such that for all $i \in I$, $M(X, \bar{\succ}_i) = M(X, \hat{\succ}_i)$. By Proposition 1, for all $i \in I$, $\bar{\succ}_i, \hat{\succ}_i \in SW(X)$, $M(X, \bar{\succ}_i) = \{t(\bar{\succ}_i)\}$ and $M(X, \hat{\succ}_i) = \{t(\hat{\succ}_i)\}$. Therefore, for each $i \in I$, $t(\bar{\succ}_i) = t(\hat{\succ}_i)$. Finally, using the definition of the median choice rule, we have $f^m(\bar{\succ}) = f^m(\hat{\succ})$, which is equivalent to say that $f^m(\cdot)$ is tops-only.[9] 2

For every agent i, to declare her top on each subset of alternatives $\tilde{X}$, $t(\tilde{\succ}_i)$, is a dominant action:

Lemma 3 *If $\tilde{\succ}$ is the restriction of the order-restricted profile $\succ$ on $\tilde{X}$, then*

$$f^m(t(\tilde{\succ}_i), \tilde{x}_{-i}) \tilde{\succ}_i f^m(\tilde{x}_i, \tilde{x}_{-i})$$

for all $i \in I$, where $\tilde{x}_{-i} = \langle \tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_{i+1}, \ldots, \tilde{x}_n \rangle \in \prod_{j \neq i} \tilde{X}$ and $x_i \in \tilde{X}$.

[9] An alternative proof of Lemma 2 can be stated following Rothstein's [18] Representative Voter Theorem. Effectively, this result says that, as long as preferences are order-restricted, the preference relation of the median voter coincides with the ordering induced by the majority rule. Therefore, it must be possible to determine the chosen alternative $f^m(\tilde{\succ})$ in every subset $\tilde{X}$ by considering only the maximal set of the median voter.

PROOF. Let $\tilde{x}_{-i}$ be a given vector of *feasible* declarations (in the sense of being a vector of rational choices), where each component x_j is the choice of agent j. We call $\tilde{x}_m = med\{t(\tilde{\succ}_i), \tilde{x}_{-i}\}$. Assume that for an $i \in I$, $i \neq m$ (the median agent), there exists an alternative $\tilde{x}_i \in \tilde{X}$ such that $f^m(\tilde{x}_i, \tilde{x}_{-i}) \tilde{\succ}_i f^m(t(\tilde{\succ}_i), \tilde{x}_{-i})$. Without loss of generality assume that the true top verifies that $t(\tilde{\succ}_i) < \tilde{x}_m$.

- If $\tilde{x}_i \leq \tilde{x}_m$, then $med(\tilde{x}_i, \tilde{x}_{-i}) = med(t(\tilde{\succ}_i), \tilde{x}_{-i})$. Therefore $f^m(\tilde{x}_i, \tilde{x}_{-i}) = f^m(t(\tilde{\succ}_i), \tilde{x}_{-i})$. Absurd.

- If $\tilde{x}_i > \tilde{x}_m$, then the new median top, $\tilde{x}_m^{'}$, will be in the interval $(\tilde{x}_m, \tilde{x}_i]$. Suppose that $\tilde{x}_m^{'} \tilde{\succ}_i \tilde{x}_m$ (i.e. according to i's *true* preferences). Then, by $OR-2$, since $\tilde{x}_m^{'} > \tilde{x}_m$, for every $j > i$ we have that

$$\tilde{x}_m^{'} \tilde{\succ}_j \tilde{x}_m \qquad (*)$$

  On the other hand we have that $t(\tilde{\succ}_i) < \tilde{x}_m$. Suppose that $m < i$, then, since $\tilde{x}_m \tilde{\succ}_m t(\tilde{\succ}_i)$ (otherwise, m would have chosen $t(\tilde{\succ}_i)$) by $OR-2$ we have that for every $j > m$, $\tilde{x}_m \tilde{\succ}_j t(\tilde{\succ}_i)$, in particular for i. Absurd. Therefore $m > i$. So, going back to expression $(*)$, it follows that $\tilde{x}_m^{'} \tilde{\succ}_m \tilde{x}_m$. Absurd, because then, if we replace the declaration of m, $\tilde{x}_m$, by the alternative declaration $\tilde{x}_m^{'}$, we have that $f^m(\tilde{x}_i, \tilde{x}_m^{'}, \tilde{x}_{j \neq i \wedge j \neq m}) \tilde{\succ}_m f^m(\tilde{x}_i, \tilde{x}_m, \tilde{x}_{j \neq i \wedge j \neq m})$. That is, $\tilde{x}_{-i}$ is not feasible, contrary to our initial assumption. 2

It follows immediately the following:

Theorem 1 $f^m : [OR(X)]^n \to X$ *is strategy-proof over each* $\tilde{X} \in \chi$.

PROOF. Immediate from Lemmas 1, 2 and 3. 2

This result can be used to prove the *implementability* of f^m:

Theorem 2 *There exists a mechanism that implements* f^m *in a dominant strategy equilibrium over* X.[10]

[10] A mechanism implementing $f^m : [OR(X)]^n \to X$ is a strategic game form G with consequences in X. $G = \langle I, (A_i), \phi \rangle$ such that A_i is the set of actions available to agent $i \in I$, and $\phi : \prod_I A_i \to X$ is an outcome function. Given the preferences $\langle \succ_1, \ldots, \succ_n \rangle \in [OR(X)]^n$, $(G, \langle \succ_1, \ldots, \succ_n \rangle)$ becomes a game. Finally, given a solution concept $\mathcal{S}$, $\phi(\mathcal{S}(G, \langle \succ_1, \ldots, \succ_n \rangle)) = f^m(\succ_1, \ldots, \succ_n)$ (Osborne and Rubinstein [15]).

PROOF. To implement f^m, we have to define a game form G, find and adequate outcome function ϕ, and show that $\phi(\mathcal{D}(G, \langle \succ_1, \ldots, \succ_n \rangle)) = f^m(\succ_1, \ldots, \succ_n)$, where $\mathcal{D}$ yields the dominant strategy equilibria of the game. Let us assume that $A_i = X$, for each $i \in I$. Then, an action by agent i is to choose an element of X. The outcome of the game is $\phi(x_1, \ldots, x_n) = med\{x_i\}$; that is, the median of the choices of the agents. Consider the action profile $\langle t(\succ_1), \ldots, t(\succ_n) \rangle$. We will show that it constitutes a dominant strategy equilibrium. That is,

$$\phi(x_1, \ldots, t(\succ_i), \ldots, x_n) \succ_i \phi(x_1, \ldots, x_i, \ldots, x_n),\ \forall i,\ x_i \neq t(\succ_i),\ x_{-i} \in \prod_{j \neq i} A_j$$

Since, by definition, $\phi(x_1, \ldots, t(\succ_i), \ldots, x_n) = med\{x_1, \ldots, t(\succ_i), \ldots, x_n\} = f^m(x_1, \ldots, t(\succ_i), \ldots, x_n)$, we can easily recast the proof of Lemma 3 to show that given any undominated profile x_{-i} for the rest of the agents, no x_i exists verifying that $\phi(x_i, x_{-i}) \succ_i \phi(t(\succ_i), x_{-i})$: Suppose that there exists such x_i. Then, calling $x_m = \phi(t(\succ_i), x_{-i})$ and $x_m^{'} = \phi(x_i, x_{-i})$, we have two cases:

1. $x_i \leq x_m$. Then $med\{t(\succ_i), x_{-i}\} = med\{x_i, x_{-i}\}$ therefore $\phi(t(\succ_i), x_{-i}) = \phi(x_i, x_{-i})$. Absurd.

2. $x_i \geq x_m$. Then the new median $x_m^{'}$ will be in the interval $[x_m, x_i]$. Suppose that $x_m^{'} \succ_i x_m$. Then, since the preferences verify $OR-2$ and $x_m^{'} > x_m$, then, for every $j > i$ we have that

$$x_m^{'} \succ_j x_m \qquad (*)$$

   On the other hand we have that $t(\succ_i) < x_m$. Suppose that $m < i$, then, since $x_m \succ_m t(\succ_i)$ by $OR-2$ we have that for every $j > m$, $x_m \succ_j t(\succ_i)$, in particular for i. Absurd. Therefore $m > i$. So, going back to expression $(*)$, it follows that $x_m^{'} \succ_m x_m$. Absurd, because we assumed that x_{-i} was undominated.

Therefore, $\langle t(\succ_1), \ldots, t(\succ_n) \rangle$ is a dominant strategy equilibrium, and by definition $\phi(t(\succ_1), \ldots, t(\succ_n)) = med\{t(\succ_1), \ldots, t(\succ_n)\} = f^m(t(\succ_1), \ldots, t(\succ_n))$. 2

It follows immediately that this result extends to every subset of X:

Proposition 2 *For every $\tilde{X} \in \chi$, $f^m(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) = M(\tilde{X}, \succ_m)$.*

PROOF. According to Theorem 2, there exists a mechanism that implements f^m over X. This mechanism assumes that each agent chooses an element of X. If we restrict the mechanism to $\tilde{X}$, by Lemma 1 we have that preferences remain order-restricted on this subset, therefore the argument of Theorem 2 applies also for f^m over $\tilde{X}$. Since f^m yields a dominant outcome for each agent, we have that this is also true for each $\tilde{X}$. That is, $\tilde{G}$, where each $\tilde{A}_i = \tilde{X}$, and $\tilde{\phi} \equiv \phi$, implements f^m over $\tilde{X}$, yielding $\langle t(\tilde{\succ}_1), \ldots, t(\tilde{\succ}_n) \rangle$ as a dominant strategy equilibrium. Moreover, $f^m(t(\tilde{\succ}_1), \ldots, t(\tilde{\succ}_n)) = \tilde{\phi}(t(\tilde{\succ}_1), \ldots, t(\tilde{\succ}_n)) = t(\tilde{\succ}_m)$, where m is the median agent in I. 2

4 Discussion

This paper analyzes the existence of non-trivial strategy-proof social choice rules on the domain of order-restricted preferences. That is, it studies strategy-proofness on a preference domain where there exists a natural ordering of the individuals, rather than of the alternatives. Hence, at least in this point, the approach followed here differs from most of the related literature and therefore may be considered as a complete original work.

Besides, with respect to the relevance of this research, we simply emphasize that the preference domain taken into consideration in the analysis has importance in many branches of economic analysis. As it has been proven recently, order-restriction is connected with the more simple and economically intuitive property of ordinal single-crossing, which is invoked frequently in several applications of voting theory, mechanism design, principal-agent theory and information economics. All this by its own justify the analysis of strategy-proofness on that preference domain.

Regarding to the results, the paper shows that the so-called median choice rule is strategy-proof not only over the full set of alternatives, but also over every possible policy *agenda* (Theorem 1). This finding, which follows from the *invariance* of order-restriction in the real line (Lemma 1), looks particularly appealing for models of political economy, where voters usually have to choose from sets with few policies rather than from the full set of alternatives.

Furthermore, it contrasts with the results found by Barberá and Jackson [5] and Barberá et at. [6], on the domain of single-peaked preferences.

Effectively, both papers show that social choice rules, which satisfy strategy-proofness over the full set of alternatives, may fail to verify this property over arbitrarily restricted subsets. In that settings, problems arises from the fact that single-peakedness, unlike order-restriction, does not restrict too much the direction of preferences among alternatives that are not top; that is, because single-peakedness allows a great amount of heterogeneity, across individuals, between those alternatives that are not top-ranked.

As a by-product, this article also deals with the problem of the implementability in dominant strategies of Rothstein's [18] Representative Voter Theorem. This theorem is important because it offers a formal justification for a common analytical technique, frequently used in collective decision-making processes with multiple and heterogeneous voters. This technique consists of reducing the constituency to a single representative voter, or of considering the electorate as a whole "homogeneous" individual.[11]

As we said before, the justification for such procedure resides in Rothstein's Representative Voter Theorem. But, unlike the median one, whose non-cooperative strategic foundation was provided by Moulin [14], this result is based on the assumption of *sincere voting*. That is, it is based on the hypothesis that in every election each citizen votes for the alternative that gives him the highest utility according to his policy preferences.

Obviously, this assumption is hard to maintain when the study focuses on policy choices taken in game-theoretic frameworks. Conversely, following the general perspective that votes are resources and that voters would like to do the best use of these resources, in such strategic environments a more natural assumption is that players are *forward-looking* or *sophisticated* voters. Thus, an immediate question is whether or not the Representative Voter Theorem continues to apply in those models. Fortunately, Theorem 2 provides a positive answer to this problem.

Finally, this paper suggests also a number of new questions, such as the general form (i.e., the complete class), if any, of strategy-proof social choice rules on the domain of order-restricted preferences; the extent to which we can relax order-restriction and maintain at the same time strategy-proofness; the kind of information that is sufficient to demand from individuals if social choice rules are strategy-proof and preferences order-restricted; etc. We hope to deal with these and with more problems in future research.

[11] For example, in the applications of the agenda-setter model, among the papers that employ this approach are Ingberman [12], Banks [3] [4], Baldez and Carey [2], etc.

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