TRANSACTIONALLY EFFICIENT MARKETS, DYNAMIC ARBITRAGE AND MICROSTRUCTURE

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ABSTRACT

In this paper, we introduce a Transactionally Efficient Market Model, which evolves from the standard efficient market model, encompassing both transaction costs and bid-ask prices. Hence, we delve into how arbitrage makes its way within this complex setting. The main outgrowth of the analysis is the "trap set", which is the place where most of price trajectories should enter to put an end to supernormal profits, although the underlying dynamics seems far from coming to a halt, and becomes bewildering instead. Bid-ask arbitrage gaps will prove useful to track down those adjustments of current prices, transaction costs and fundamental values. At this point, we define a transactionally efficacious market. Furthermore, a non linear dynamics whose environment gives room to mediator and microstructure, will lead us to prove the existence of a vectorial arbitrage gap mapping which becomes operational at managing the transactional efficiency of the market, in a complex surroundings with chaotic patterns eventually. Summing up: transactionally efficient markets are those markets which are informative efficient and transactionally efficacious.

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Key Words: Dynamic Arbitrage; microstructure; transactional efficiency; chaos.

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1.- INTRODUCTION

For the last fifteen years, market microstructure and transaction costs have become a provocative issue in Finance. Not surprisingly, an outstanding literature has drawn attention to disequilibrium values, transaction prices and model values. Developments running along this path has been surveyed by O'Hara (1997). Papers as those by Harris (1998) and Huang-Stoll (1999), as well the latest book on microstructure by Spulberg (1999) do clearly stand out on nurturing further research.

This innovative point of view will be blended with Complex Dynamics. Although research on this matter is a well established academic field, it seems to be only at the outset either in Finance or in Economics. Fitting references for mathematical foundations are found in Hubbard-Hubbard (1999) and Devaney (1989). Seminal work has been carried out by Professor Day (1993, 1994, 1995). Chaotic Dynamic Systems are treated in Apreda (1997); an alternative approach to this paper's contents, grounded on a multiplicative model is to be found in Apreda (1998-b, 1999). Recent work in non-linear dynamics applied to Finance is surveyed in Cuthbertson (1996).

What we want to do in this paper can be broken down into the following stages:

- Firstly, we frame transaction costs and bisd-ask prices into arbitrage strategies made out of boundary conditions, trading rules and balances, and arbitrage dynamics. The main outgrowth is the concept of transactionally efficacious markets.

- Next, current prices, fundamental values and transactions costs are embedded in a pair of arbitrage gaps which become functional to price trajectories converging towards a "trap set", a location that prices should approach so as to preclude mispricing from taking place. From this analysis we draw the transactionally efficient market model.

- Furthermore, an adjustment price process is taken into account, and a non linear dynamic environment is devised so as to cope with microstructure and mediator activity. Complexity also may lead prices to chaotic patterns, which amounts to impredictable behavior stemming from a deterministic model. This might come in handy to the purpose of prospective empirical research.

2.- EFFICIENT MARKETS

The prevailing format for the Efficient Market Model run as follows: a market for financial assets is said to be efficient if assets prices fully reflect all available information, thus eliminating knowable opportunities for supernormal profits. In fact, this model lays on two assumptions (Begg, 1982):

- Expectations are rational so that individuals avoid forecasting errors given current information.

- Any discrepancy between the expected rates of return of different assets is quickly arbitraged to eliminate expected supernormal profits.

Remark:

As Grossman-Stiglitz pointed out (1980) in their classical paper, it is the possibility of obtaining supernormal profits in the course of arbitraging which provides the incentive to collect and process new information. The concept of arbitrage, as it is used in standard Financial Economics is reviewed in Varian (1987).

Formally, we will denote current prices at moment "**t**" with **P(t)**, and values from a valuation model at moment "**t**" with **W(t)**. We choose the following Efficient Market Model format as suitable to our purposes.

Definition 1*:*

We are going to regard a financial market as efficient if the following features hold true:

[1] Rational Expectations*:*

$$P(t+1) = E[\, P(t+1) \mid W_t \,] + e(t+1)$$

$$E[\, e(t+1) \mid W_t \,] = 0$$

$$E[\, e(t+1) \,.\, W_t \mid W_t \,] = 0$$

[2] Arbitrage Dynamics:

if | P(t) - W(t) | ³ a , with a > 0, then lim P(t) = W(t) whenever t ® t + m

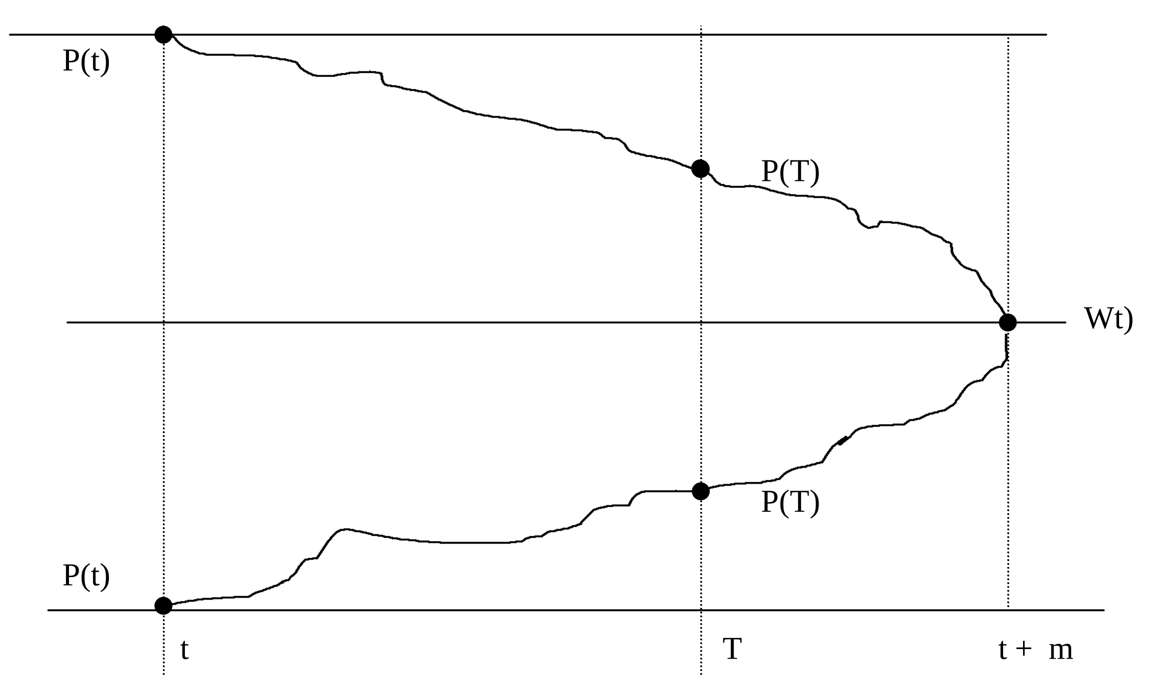
Remarks:

- Ω_t stands for an information set at moment "**t**". It is supposed that at "**t + m**" the adjustment process may have already finished. **ε(t+1)** stands for error as usual**.**
- In general **W(t) ≠ P(t)**, because of trading conditions which would trigger off arbitrage, whenever is possible. We acknowledge helpful comments on this issue from Dr. Zablotsky (The University of Cema). We are going to introduce transactions costs and bid-ask prices in **Section 4**.

The price adjustment that comes up within this environment can be traced on ***Picture 1*** below. At moment "**t**" there are arbitrage opportunities. Either **P(t)** is higher than the model value **W(t)** and then arbitrage will press down the current price, or **P(t)** is lower and such underpricing will be narrowed down up to fade away by arbitraging at time "t+m". Profits could still be collected at moment "**T**", whereas arbitrage should not longer become interesting at moment "**t+m**". The more efficient the market, the sooner "**t+m**" ought to take place.

Valuable as this approach may be, and it has been a landmark in the development of the Financial Economics, it fails to provide understanding whenever bid and ask prices are introduced and the transactional structure is taken into account. To put it briefly, we want to explore how far we should go adapting the model to this transactional framework and whether we can resourcefully model the underlying arbitrage dynamics.

Picture 1: ARBITRAGE DYNAMICS IN THE STANDARD MODEL



How do we deal with **[1]** as it is conveyed in the standard format? We could take advantage of valuation models like CAPM, for instance, to forecast the financial asset expected return, and from this to assess the expected price value for "**T**", by solving:

Remark:

We should bear in mind that W(t) embeds all the future expected cash flows that are relevant to the asset. Instead, the expected value of the asset return is a single period value.

[3]

$$\mathbf{E[\,P(t+1)\,|\,\Omega_t\,]\ /\ <\ 1\ +\ E[\,R(t,t+1)\,|\,\Omega_t\,]\ >\ \ =\ \ W(t)}$$

Currently, and mainly in textbooks, a stronger approach to the Efficient Market Hypothesis underlines informational efficiency in this way:

[4]

$$E[\,P(t+1)\mid \Omega_t\,] \;/\; <\; 1 + E[\,R(t,t+1)\mid \Omega_t\,] \;> \;=\; P(t)$$

That is to say, in a pure informationally efficient market, instead of adopting the model value **W(t)** in **[1]**, as it seems consistent with our format, the current equilibrium price is used. The main reference is Fama (1970, 1991). Nevertheless, the left side of **[3]** does strictly stem from a theoretical value.

There is a deep link between the current price and the model value, as it is deployed by **[2]** . An interesting survey about efficient markets can be found in Ball (1995).

3.- TRANSACTION COSTS

Departing from the standard model, we are going to deal with economic agents who sell or purchase their financial assets in the capital market, that is to say, through dealers. As from now

$$P(t,l) \quad , \quad P(t,s)$$

stand for purchasing and selling prices from the investor's balances, building up long or short positions, respectively. But we have to bear in mind that the following relationships hold true, from the dealer's own balances:

$$P(t,l) \;=\; P_d(t,s) \;=\; \text{ask price}$$

$$P(t,s) \;=\; P_d(t,l) \;=\; \text{bid price}$$

That is to say, investors buy assets to dealers at

$$P_d(t,s)$$

In other way, dealers sell investors assets at that price. Likewise, investors sell assets to dealers at

$$P_d(t,l)$$

which amounts to the price at which dealers are ready to buy assets.

Either by selling or buying assets investors face transaction costs,

$$K(t, l) \quad , K(t, s)$$

These transaction costs deploy a rich structure and include the following components:

- Mediator's spread and transaction completion charges (costs of submitting "limit" or "market orders", and the costs of executing such orders).
- Stamp taxes and brokerage fees.
- Cost-of-carry for long or short positions either from loans to buy assets, or from marginal charges in dealers accounts to make short selling available.
- Costs of hedging future long or short positions by using derivatives.
- Costs of search.
- Capital and Income taxes.

Remark:

- Jeffrey Roberts (1997) stressed the role of taxes as transaction costs. Also, Damodaran (1997) underlined a pragmatic analysis about the "hidden costs of trading".

4.- ARBITRAGE DYNAMICS WITH TRANSACTION COSTS

When we try to expand the standard model to make it inclusive of transaction prices and costs, we have to split up the arbitrage process into boundary conditions, trading rules, and arbitrage dynamics inclusive of transaction costs. In this within this new surroundings that the investor not only takes prices from dealer's current quotations but he also takes advantages of valuation models to figure out **W(t)** values for financial assets. He can set up either long or short positions when matching the current prices with fundamental values **W(t).** These alternatives are worthy of further analysis.

- ***ARBITRAGE STRATEGY 1: LONG POSITION***

The intuition under this strategy amounts to taking profit whenever mispricing chances arise. In case the valuation model were correct, a purchase price (after transaction costs) still below the fundamental value, it may certainly increase to such extent that selling it later might be advantageous.

[5] **Boundary Condition:** $P(t, l) + K(t,l) < W(t)$

Trading Rule: hold long

Arbitrage Dynamics: $P(t,l) - W(t) + K(t,l) \rightarrow 0$ as long as $t \rightarrow t + m$

- ***ARBITRAGE STRATEGY 2: SHORT POSITION***

Here, a symmetric intuition to that of strategy 1 holds true.

[6] **Boundary Condition:** $\mathbf{P(t, s) - K(t,s) > W(t)}$

Trading Rule: hold short

Arbitrage Dynamics: $\mathbf{P(t,s) - W(t) - K(t,s) \rightarrow 0}$ as long as $t \rightarrow t + m$

As we can see, the standard efficiency model is a very single case because it doesn't regard transaction costs as a fact of life and so arbitrage takes place between what is a notional price (either the average bid-ask prices or a current price which is supposed to be in equilibrium) and a model value. But if we plug into the arbitrage process relations **[5]** and **[6]** , then two benchmark values become unavoidable:

Boundary conditions for prices:

$$\begin{cases} \mathbf{W(t) - K(t,l)} \\ \mathbf{W(t) + K(t,s)} \end{cases}$$

Picture 2 displays one likely pattern for the process. The full line paths depicting the trajectories of long and short prices are only one among other likely stylized paths. The important fact, however, is that for arbitrage to take place transaction costs need to get covered.

Transaction costs coverage

$$\begin{cases} \mathbf{P(T,s) - P(t,l) > K(t,s) + K(t,l)} \\ \mathbf{P(t,s) - P(T,l) > K(t,s) + K(t,l)} \end{cases}$$

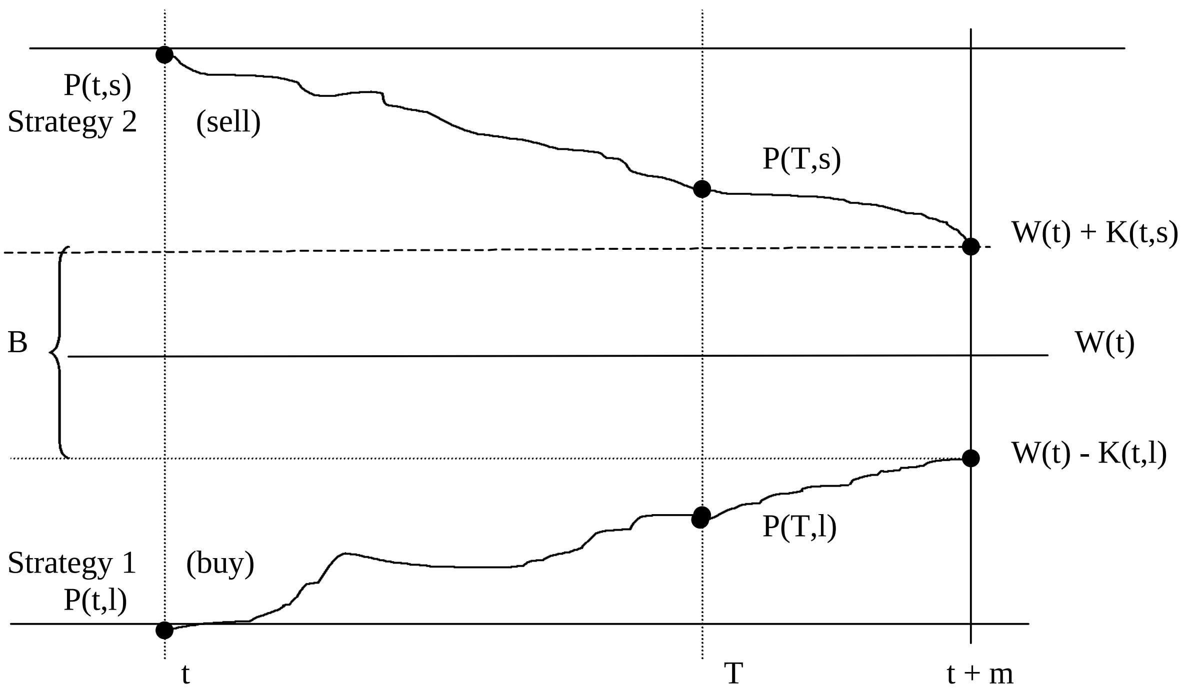
<u>Summing up</u>:

If we expand the standard efficient market model so as to include transaction prices and costs, then informationally efficiency will require that prices converge this way:

[7]

$$\begin{cases} \mathbf{P(t,s) \rightarrow W(t) + K(t,s)} \text{ , as long as } t \rightarrow t + m \\ \mathbf{P(t,l) \rightarrow W(t) - K(t,l)} \text{ , as long as } t \rightarrow t + m \end{cases}$$

Picture 2: ARBITRAGE DYNAMICS WITH TRANSACTION COSTS



Criticism:

In a mediator setting, informationally efficiency alone is not enough, because the arbitrage dynamics comes up as a very complex process. And it is a striking fact, as we draw it from ***Picture 2,*** that we have to cope with a positive real numbers closed interval

$$\mathbf{B} = [\ \mathbf{W(t) - K(t,l)}\ ;\ \mathbf{W(t) + K(t,s)}\ ]$$

$$\mathbf{B} = \{\ \mathbf{x} \in \mathbf{R}^{+} : \mathbf{W(t) - K(t,l)} \leq \mathbf{x} \leq \mathbf{W(t) + K(t,s)}\ \}$$

This set establishes a strip of values which becomes complex and functional for transactions to become efficacious.

Whitin this “trap set”, prices would prevent any arbitrage transaction from taking place because the cost of performing them might not even have been covered. Therefore, it seems worth qualifiying the “trap set” with a definition, and searching for more details on this issue.

<u>Definition 2:</u>

We wil call "the trap set" to the set

$$B = \{ x : W(t) - K(t,l) \quad £ \; x \quad £ \quad W(t) + K(t,s) \}$$

5.- THE TRAP SET DYNAMICS

The trap set leads us to a pair of distinctive problems:

- What would happen if a price lay within the trap set?
- Could a price lie within **B** but the arbitrage proceed as long as expectations of profit when reverting the position were big enough so as to cover transaction costs?

To cope with the first question, we have to discuss long and short spot positions. Then we need to deal with expected future prices matching the spot positions. Last of all, we are going to delve into the issue of convergence of bid and ask prices to the trap set.

a) LONG AND SHORT SPOT POSITIONS

- ***<u>Case 1: Long Positions</u>***

$$\textbf{if } P(t,l) \in B \quad \Rightarrow \quad P(t,l) \geq W(t) - K(t,l) \quad \Rightarrow \quad P(t,l) + K(t,l) \geq W(t)$$

against **[5]** and arbitrage is, therefore, ruled out.

- ***<u>Case 2: Short Positions</u>***

$$\textbf{if } P(t,s) \in B \quad \Rightarrow \quad P(t,s) \leq W(t) + K(t,s) \quad \Rightarrow \quad P(t,s) - K(t,s) \leq W(t)$$

against **[6]** and arbitrage is ruled out too.

That is to say, for arbitrage to start at moment "**t**", prices ought to lie outside the trap set **B**: long prices below the lower point of **B**; short prices above the upper point of **B**. However, **Case 1** and **Case 2** don´t preclude real transactions from taking place. In fact, within the rap set B we either buy or sell, perhaps at a loss.

b) SPOT AND FUTURE POSITIONS

It's time to address the second problem outlined at the beginning of this section, whether a price could lie within **B** but the arbitrage proceed as long as expectations of profit when reverting the position were big enough so as to cover transaction costs. Therefore, we need to analyse the whole arbitrage process as it comes out of a trading off between spot prices and expected futures ones.

We must find out boundary conditions for future prices, at the moment "**T**". Only for the time being, we keep the same assumptions that **W(t)**, **K(t,s)**, **K(t,l)** are not to change.

- ***<u>LONG POSITION AT "t"; SHORT POSITION AT "T"</u>***

At "**T**", we would reverse the long position to get a profit whenever:

$$\{ E[P(T,s)] - K(T,s) \} - \{ P(t,l) + K(t,l) \} > 0$$

In fact, we wrote the expected value as **E[P(T,s)]** to make notation simpler. However, we must bear in mind that we intend to mean a conditional expectation onto the information set at moment "**t**".

$$E[P(T,s) \mid \Omega_t]$$

Two situations must be examined:

- ***<u>Situation 1</u>*:** $E[P(T,s)] - K(T,s) \geq W(t)$

But it is by **[5]** that: $P(t,l) + K(t,l) < W(t)$, then hold long,

hence:

$$\{ E[P(T,s)] - K(T,s) \} - \{ P(t,l) + K(t,l) \} > 0$$

It seems to be the most desirable trading setting, because both **E[P(T,s)]** and **P(t,l)** lie outside the trap set **B,** and we get a sure profit.

- ***<u>Situation 2</u>*:** $E[P(T, s)] - K(T,l) \leq W(t)$

Then

$$\{ E[P(T,s)] - K(T,s) \} - \{ P(t,l) + K(t,l) \} < W(t) - \{ P(t,l) + K(t,l) \} = A$$

and the right side is a positive number **A**, by **[5]**. Hence:

$$\{ E[P(T,s)] - K(T,s) \} - \{ P(t,l) + K(t,l) \} < A$$

In this way the left side can become negative, with no granted profit. Why might this happen? This takes place when **E[P(T,s)]** falls down into the trap set or, further still, one of the prices lies outside **B**, the other within it, but the gap between them is not wide enough to cover transaction costs. That is to say:

$$| E[P(T,s)] - P(t,l) | < | K(T,s) + K(t,l) |$$

<u>Conclusion</u>:

In order to start a long position at moment "t", it is expected for

neither P(t,l) nor E[P(T,s)]

to lie within the trap set **B.** When one of them lies in B, the other must stay outside B granting a spread between them just to cover transaction costs.

- ***<u>SHORT POSITION AT "t"; LONG POSITION AT "T"</u>***

Similarly as we did above, the analysis would lead us to the following conclusion: in order to start a short position at moment "t", it is expected for neither

P(t,s) nor E[P(T,l)]

to lie within the trap set **B.** When one of them lies in **B**, the other must stay outside **B** granting a spread between them just to cover transaction costs.

c) CONVERGENCE TO THE TRAP SET

From the foregoing analysis in a) and b), we can draw some consequences:

- In order to prevent arbitrage, both **P(t,l)** and **P(t,s)** must go into the trap set. That is to say:

$$P(t,l) \rightarrow B$$

$$P(t,s) \rightarrow B$$

Hence, prices converge towards a set and remain trapped within it. And this may take place if and only if

$$\begin{cases} \mathbf{P(t,l)} > \mathbf{W(t) - K(t,l)} & \text{as long as } t \rightarrow t + m \\ \mathbf{P(t,s)} < \mathbf{W(t) - K(t,s)} & \text{as long as } t \rightarrow t + m \end{cases}$$

- Trading should go on whenever transaction costs be covered
- It is no longer necessary as it should be expected in the standard efficient model framework, that

$$\begin{cases} \mathbf{P(t,s)} \rightarrow \mathbf{W(t) + K(t,s)} & \text{, as long as } t \rightarrow t + m \\ \mathbf{P(t,l)} \rightarrow \mathbf{W(t) - K(t,l)} & \text{, as long as } t \rightarrow t + m \end{cases}$$

- In accordance with **[8]** and **[9]**, instead of point-wise convergence, we should expect set-wise convergence.

6.- A TRANSACTIONALLY EFFICIENT FINANCIAL MARKET

As from now we are ready to remove the assumptions:

K(t,s) = constant; K(t,l) = constant; W(t) = constant

Taking advantage of the foregoing analysis, for an efficient market to perform smoothly inclusive of transaction costs the following arbitrage strategies should hold: (Look at ***Picture 2*** above).

- ***ARBITRAGE STRATEGY 1: LONG SPOT + SHORT FUTURE POSITIONS***

[8] **Boundary Condition:** $\mathbf{P(t, l) + K(t,l) < W(t)}$

Trading Rule: hold long at "t" and short at "T"

Trading Balance: $\mathbf{[\,P(T,s) - K(T,s)\,] - [\,P(t,l) + K(t,l)\,] > 0}$

Arbitrage Dynamics: $\mathbf{P(t,s) \rightarrow B}$ **as long as** $\mathbf{t \rightarrow t + m}$

- ***ARBITRAGE STRATEGY 2: SHORT SPOT + LONG FUTURE POSITIONS***

[9] **Boundary Condition:** $\mathbf{P(t,s) - K(t,s) > W(t)}$

Trading Rule: hold short at "t" and long at "T"

Trading Balance: $\mathbf{[P(t,s) - K(t,s)] - [P(T,l) + K(T,l)] > 0}$

Arbitrage Dynamics: $\mathbf{P(t,l) \rightarrow B}$ **as long as** $\mathbf{t \rightarrow t + m}$

The foregoing strategies allow for the market to become transactionally efficacious.

Definition 3:

A market of financial assets is said transactionally efficaciuos

whenever both Arbitrage Strategy 1 and 2 hold always true.

Now we are ready to cope with transactional efficiency.

Definition 4:

A market of financial assets will be transactionally efficient *whenever the following conditions are met:*

[10] ***Rational Expectations****:*

$$P(t+1, l) = E[P(t+1, l) \mid W_t] + e(t+1, l)$$

$$P(t+1, s) = E[P(t+1, s) \mid W_t] + e(t+1, s)$$

$$E[e(t+1, l) \mid W_t] = E[e(t+1, s) \mid W_t] = 0$$

$$E[e(t+1, l) . W_t \mid W_t] = E[e(t+1, s) . W_t \mid W_t] = 0$$

$$E[e(t+1, l) . e(t+1, s) \mid W_t] = 0$$

[11] ***Arbitrage Strategies [8] and [9] hold always true.***

That is to say, a transactionally efficient market brings together informative efficiency and is transactional efficacy.

It seems advisable to give a step further and build up a trap set with an explicit transaction costs structure.

7.- ARBITRAGE CONTROL FUNCTIONS

Next diagram comes out from the former one, after changing the horizontal benchmark (which it was **W(t)** = constant) by a new one (**0 = W(t) - W(t)**). The other horizontal lines catch up with this change by substracting **W(t)** in all cases. (***Picture 3***).Such a new coordinates system leads to another trap set **B*.** Hence, instead of working with the interval:

$$\mathbf{B = [\ W(t) - K(t,l)\ ;\ W(t) + K(t,s)\ ]}$$

we shall adopt

$$\mathbf{B^* = [\ -K(t,l)\ ;\ K(t,s)\ ] = \{\ x : -K(t,l) \le x \le K(t,s)\ \}}$$

Points in the former **B** were prices, points in the new **B*** set are shaped as (**P(t) - W(t)**). Besides, we define the following arbitrage control functions on a real variable "**t**".

[12]

Arbitrage Control Functions:

$$\begin{cases} \beta(t,l) = P(t,l) - W(t) + K(t,l) \\ \beta(t,s) = P(t,s) - W(t) - K(t,s) \end{cases}$$

We should bear in mind that **β(t, l) and β(t, s)** measure whether arbitrage pays off or not. When either **β(t,l)** is negative, or **β(t,s)** is positive, arbitrage makes sense.

Now, we can translate the **Strategies 1** and **2** in terms of arbitrage control functions (see ***Picture 3*** on next page):

- ***ARBITRAGE STRATEGY 1: LONG SPOT + SHORT FUTURE POSITIONS***

[13]

$$\beta(t,l) = P(t,l) - W(t) + K(t,l) < 0$$

$$\beta(T,s) = P(T,s) - W(T) - K(T,s) > 0$$

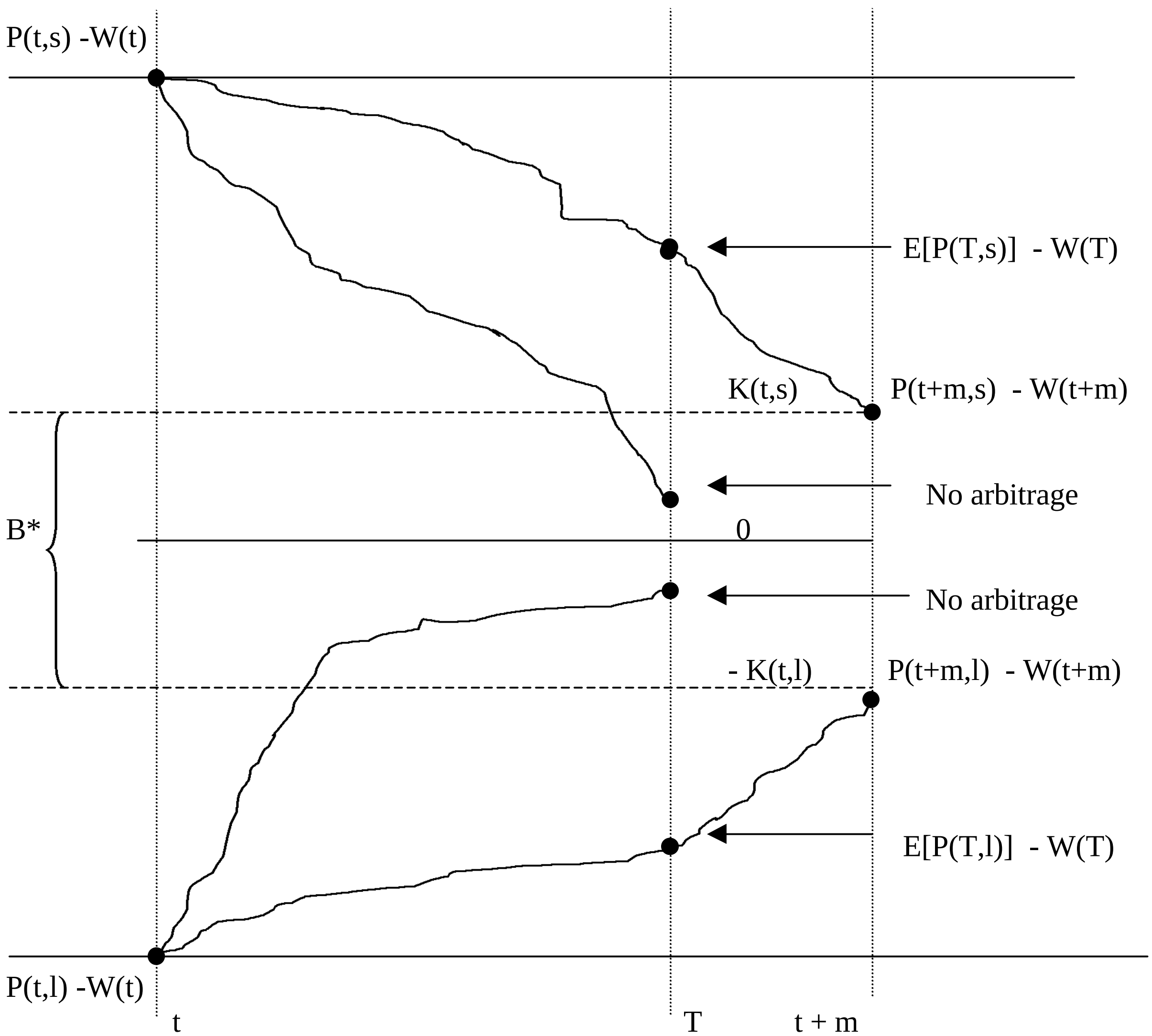
- ***ARBITRAGE STRATEGY 2: SHORT SPOT + LONG FUTURE POSITIONS***

[14]

$$\beta(t,s) = P(t,s) - W(t) - K(t,s) > 0$$

$$\beta(T, l) = P(T,l) - W(T) + K(T,l) < 0$$

Picture 3: ARBITRAGE DYNAMICS IN THE TRANSACTIONALLY EFFICIENT MARKET MODEL



Taking advantage of **[14]** , we can suggest the following vectorial function

[15]

$$\boldsymbol{\beta(t)} = \begin{pmatrix} \boldsymbol{\beta(t,l)} \\ \boldsymbol{\beta(t,s)} \end{pmatrix} = \begin{pmatrix} \mathbf{P\,(t,l) - W(t) + K(t,l)} \\ \mathbf{P\,(t,s) - W(t) - K(t,s)} \end{pmatrix}$$

that allows for the following definition.

<u>Definition 5:</u>

The vectorial function

$$\boldsymbol{b(t)} = \begin{pmatrix} \boldsymbol{b(t,l)} \\ \boldsymbol{b(t,s)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{P\,(t,l) - W(t) + K(t,l)} \\ \boldsymbol{P\,(t,s) - W(t) - K(t,s)} \end{pmatrix}$$

it will be called the Arbitrage Dynamic Gap

We see that **β(t)** has a complex behavior as regards arbitrage expectations. Besides, convergence holds true in a transactionally efficient market. But convergence here, it means convergence towards the "trap set". Therefore,

Arbitrage Expectations:

$$\begin{cases} \mathbf{P(t,l) \rightarrow B^* \quad as\ t \rightarrow t + m} \\ \mathbf{P(t,s) \rightarrow B^* \quad as\ t \rightarrow t + m} \end{cases}$$

<u>Remark:</u>

- This vectorial function not only allows for changes in the model values **W(t)**, but in transaction costs functions either.

In order for the transactionally efficient market to become instrumental, we need to address three issues:

- What would it happen to **β(t)** behavior as soon as microstructure came up into this world?
- How would **β(t)** answer whenever a non linear dynamics for prices were deployed?

- How would $\beta(t)$ perform through a price adjustment process?

8.- PRICE ADJUSTMENT BEHAVIOR

A mediator, or professional trader, buys financial assets to suppliers at a price

$$P_d(t,l)$$

which can be sold at time "t", at price

$$P_d(t,s) = (1 + \nu(t)) \cdot P_d(t,l)$$

where "$\nu(t)$" stands for the mediator's mark-up.

So as to proceed to our main outcome, we have to distinguish two stages. In the first one, a dynamic adjustment for bid-ask prices takes place eventually. In the second one, a complex environment inclusive of microstructure and mediator will be brought into view.

- ***FIRST STAGE :***
DYNAMIC ADJUSTMENT IN BID AND ASK PRICES

Let us consider the following adjusting dynamical settings:

[16]

$$\begin{cases} dP(t,l)/dt = g_1[P(t,l) - W(t) + K(t,l)] \\ dP(t,s)/dt = g_2[P(t,s) - W(t) - K(t,s)] \end{cases}$$

where g_1, g_2 are increasing monotonous functions.

Bearing in mind what was told in **Section 3** about the relationship between prices from the investor's balance side and prices from dealer's balance side:

$$P(t,l) = P_d(t,s) = \text{ask price}$$

$$P(t,s) = P_d(t,l) = \text{bid price}$$

We can rewrite **[16]** this way:

$$\begin{cases} dP_d(t,s)/dt = g_1[P_d(t,s) - W(t) + K(t,l)] \\ dP_d(t,l)/dt = g_2[P_d(t,l) - W(t) - K(t,s)] \end{cases}$$

This sort of price adjustment come across the discrepancy between current prices, blending model values and transaction costs, as measured by the dynamic arbitrage gap. That is to say,

[17]

$$\begin{cases} dP_d(t,s)/dt = dP(t,l)/dt = g_1[\beta(t,l)] \\ dP_d(t,l)/dt = dP(t,s)/dt = g_1[\beta(t,s)] \end{cases}$$

Next, we linearize with the help of a Taylor expansion,

[18]

$$\begin{cases} dP_d(t,s)/dt = a_1 \cdot \beta(t,l) \\ dP_d(t,l)/dt = a_1 \cdot \beta(t,s) \end{cases}$$

Remark:

It is supposed that the excess demand is Samuelson-type, that is to say, prices change as a monotonically increasing function of excess demand. Besides, demand and supply performs as in the so called normal markets, with derivatives $g_1' > 0$; $g_2' > 0$. Further details in Day (1993).

• *SECOND STAGE: COMPLEX ENVIRONMENT INCLUDING MICROSTRUCTURE AND MEDIATOR*

Starting with an excess demand

[19]

$$\varepsilon[P_d(t,l), P_d(t,s)] = D(P_d(t,s)) - S(P_d(t,l))$$

and by making next the mediator spread explicit:

[20]

$$\varepsilon[P_d(t,l), P_d(t,s)] = D((1+\nu(t))\,P_d(t,l)) - S(P_d(t,l))$$

we sensitize the excess demand to supply and demand shifts, calling for a parametric pattern as

[21]

$$\varepsilon(P_d(t,l), P_d(t,s), \mu) = \mu \cdot \varepsilon(P_d(t,l)) = \mu \cdot [D(P_d(t,s)) - S(P_d(t,l))]$$

taking any deviation from the benchmark $\mu = 1$, as a market's breadth measure (or the extent of the market). This embeds the excess demand in a market microstructure setting.

Remarks:

- In this section, we have profited from Apreda (1998-a, 1998-b). By the way, **[21]** is very simple indeed, because one of our main goals here is to prove there is a general pattern on non linear behavior inclusive of microstructure and transaction costs features.
- Spread-modelling is extensively developed in Huan-Stoll (1997). For microstructure stochastic models O'Hara conveys a deep discussion on this subject. The most updated approach seems to be Spulberg (1999)
- Lately, a paper by Harris (1998) showed how to alternatively deal with Microstructure by means of a dynamic programming framework and derived optimal dynamic order submission strategies. This should be an interesting approach worthy of further research, just to develop connections with dynamical deterministic systems.

We are to suppose that prices adjustment develop from the following framework:

[22]

$$\mathbf{P_d(t+1, l) - P_d(t, l) = \lambda(l) . \varepsilon[P_d(t, l), P_d(t, s), \mu]}$$

$$\mathbf{P_d(t+1, s) - P_d(t, s) = \lambda(s) . \varepsilon[P_d(t, l), P_d(t, s), \mu]}$$

It is an unmistakable consequence from **[21]** that changes in prices come from excess demand with respect to spread structure, adjustment velocity in terms of excess demand and, finally, the breadth of the market. This embeds the price change in a market microstructure setting and takes into account transaction costs directly related to the mediator.

Remarks:

The determination of $\lambda(l)$ lays on these grounds:

a) Firstly, the increase in the supply of the financial asset translates an inventory change:

$$\mathbf{s(t+1) - s(t) = - \varepsilon(P_d(t,l), \mu, \nu(t)) = - \mu \cdot \{ D [(1+\nu(t)) P_d(t,l)] - S(P_d(t,l)) \}}$$

b) In real life, the mediator doesn't know neither the supply-demand structure, nor the clearing price. The relevant variable here, for him, is how his own inventory changes allowing him to mark up or down his transaction price; see Day (1994). The adjustment comes up as:

$$\mathbf{P_d(t+1,l) - P_d(t,l) = \lambda(l) \cdot [s(t+1) - s(t)]}$$

Now he can estimate $\lambda(l)$, as prices change and adjustement through excess demand takes place,

$$P_d(t+1,l) - P_d(t,l) = \lambda(l) \cdot \varepsilon\{ P_d(t,l), P_d(t,s),), \mu, \nu(t) \} \quad ; \quad \lambda > 0$$

c) On the other hand,

$$P_d(t+1,s) - P_d(t,s) = (1+\nu(t)) \cdot [P_d(t+1,l) - P_d(t,l)]$$

$$P_d(t+1,s) - P_d(t,s) = (1+\nu(t)) \cdot \lambda(l) \cdot \varepsilon\{ P_d(t,l), P_d(t,s),), \mu, \nu(t) \}$$

At last, we can take

$$\lambda(s) = (1+\nu(t)) \cdot \lambda(l)$$

Preventing negative prices, we arrive at the expected price as function of the breadth of the market μ, adjustment velocities $\lambda(l)$ and $\lambda(s)$, mark-up $\nu(t)$ and the excess demand:

[23]

$$P_d(t+1, l) = \theta_1 (P_d(t, l))$$

$$= \mathrm{Max}\{ 0, P_d(t, l) + \mu \cdot \lambda(l) \cdot \{ D[(1+\nu(t)) P_d(t, l)] - S(P_d(t,l)) \}\}$$

$$P_d(t+1, s) = \theta_2 (P_d(t, s))$$

$$= \mathrm{Max}\{ 0, P_d(t, s) + \mu \cdot \lambda(s) \cdot \{ D[(1+\nu(t)) P_d(t, l)] - S(P_d(t, l)) \}\}$$

Let us watch **[23]** closely. We have here two price functions, each of them defined on positive real numbers within an iterative framework, and a time-marker to keep record of iterations and defined in the realm of natural numbers. This amounts to a two-dimensional dynamical system:

$$< R^+, [\theta_1, \theta_2], N >$$

As long as we pick up a moment **t = t(0)**, the system evolves by iteration from the initial condition, bringing about a pair of orbits

$$\{ \theta_1^k (P_d(t, l)) \} =$$

$$\{ P_d[t(0), l], P_d[t(0)+1, l], P_d[t(0)+2, l], P_d[t(0)+3, l], P_d[t(0)+4, l] \;\ldots\ldots\ldots\; \}$$

$$\{ \theta_2^k (P_d(t, s)) \} =$$

$$\{ P_d[t(0), s], P_d[t(0)+1, s], P_d[t(0)+2, s], P_d[t(0)+3, s], P_d[t(0)+4, s] \;\ldots\ldots\ldots\; \}$$

Thus, the structure depicted in **[23]** leads to a complex deterministic system.

8.1.- STABILITY ANALYSIS

It has been proved by Day (1994) that stability analysis applied to price relations the sort of **[23]** leads to the following strong statement:

- The absolute and relative adjustement in normal markets shows every feature attending simple dynamics and complex dynamics. In general, convergence towards an only competitive equilibrium, convergence to periodic cycles, chaotic topology, strongly chaotic trajectories or self-destruction eventually.

Further, it has also been proved that for normal demand and supply functions, both mediator tatonnement and relative mediator tatonnement displays the following features:

- a unique nonstationary state $\mathbf{p}^{o}$ exists under competitive surroundings;
- for a wide range of parameters values, the process converge to a market clearing equilibrium;
- also, for a wide range of parameter values, the process exhibits cyclic or chaotic price sequences.

Further details on stability analysis can mainly be found in Day (1993, 1994). Also, in Apreda (1998-a).

9.- COMPLEX DYNAMICS FOR THE ARBITRAGE GAP

After encompassing both the dynamic adjustment behavior and the complex-dynamics with professional traders, we are ready for dealing with an existence lemma.

Lemma

For any financial asset in a transactionally efficient market with mediator and demand-supply functions as in normal market, with price adjustment as of Samuelson type:

a) there are dynamic arbitrage gaps among the bid-ask prices, the model value and the transaction costs involved;

b) which also depends, locally, on the market structure and the mediator spread;

c) besides, the dynamics is non-linear and the dynamic arbitrage gap shows complex behavior and chaotic patterns as well.

d) Lately, the dynamic arbitrage gap can spell out price adjustment.

Proof: taking advantage of **Definition 3**:

$$\beta(t) = \begin{pmatrix} \beta(t,l) \\ \beta(t,s) \end{pmatrix} = \begin{pmatrix} P_d(t,s) - W(t) + K(t,l) \\ P_d(t,l) - W(t) - K(t,s) \end{pmatrix}$$

and from **[23]**

$$P_d(t+1, l) = \theta_1 (P_d(t, l))$$

$$= \text{Max} \{ 0,\ P_d(t, l) + \mu \cdot \lambda(l) \cdot \{ D[(1+\nu(t)) P_d(t, l)] - S(P_d(t,l)) \}\}$$

$$P_d(t+1, s) = \theta_2 (P_d(t, s))$$

$$= \text{Max} \{ 0,\ P_d(t, s) + \mu \cdot \lambda(s) \cdot \{ D[(1+\nu(t)) P_d(t, l)] - S(P_d(t, l)) \}\}$$

we can plug this dynamic environment into the vectorial function **β(t).**

The stability analysis outlined at **Section 8.1** upholds that the gap behavior is non linear and within certain range of parameters values, chaotic. This applies to both functional componentes in the vectorial function:

[24]

$$\begin{pmatrix} \beta(t,l) \\ \beta(t,s) \end{pmatrix} =$$

$$\begin{pmatrix} \text{Max}\left\{0,\ P_d(t, s) + \mu \cdot \lambda(s) \cdot \{D[(1+\nu(t)) P_d(t, l)] - S(P_d(t, l)) \}\right\} - W(t) + K(t,l) \\ \text{Max}\left\{0,\ P_d(t, l) + \mu \cdot \lambda(l) \cdot \{D[(1+\nu(t)) P_d(t, l)] - S(P_d(t,l)) \}\right\} - W(t) - K(t,s) \end{pmatrix}$$

On the other hand, and providing standard regularity conditions are fulfilled, as we saw in section **7**, we can take advantage of **[17]** .

$$\begin{pmatrix} dP_d(t, s)/dt \\ dP_d(t, l)/dt \end{pmatrix} = \begin{pmatrix} a_1 . \beta(t, l) \\ a_1 . \beta(t, s) \end{pmatrix}$$

Finally, drawing from **[24],** we are led to:

$$\begin{pmatrix} dP_d(t,s)/dt \\ dP_d(t,l)/dt \end{pmatrix} =$$

$$\begin{pmatrix} \text{Max}\left\{0,\ P_d(t,s) + \mu\cdot\lambda(s)\cdot\{D[(1+\nu(t))P_d(t,l)] - S(P_d(t,l))\}\right\} - W(t) + K(t,l) \\ \text{Max}\left\{0,\ P_d(t,l) + \mu\cdot\lambda(l)\cdot\{D[(1+\nu(t))P_d(t,l)] - S(P_d(t,l))\}\right\} - W(t) - K(t,s) \end{pmatrix}$$

and this says that the dynamic arbitrage gap is locally instrumental in the price adjustment process

Remark:

If we raised the question whether we can follow the trace of the arbitrage gap along stochastic trajectories, the answer would be affirmative, as we have proved in other research paper, making use of stochastic differential equations inclusive of transaction costs; see Apreda (1998-a).

10.- CONCLUSIONS

(a) We have produced a dynamic model for prices behavior closely intertwined with the market microstructure and transaction costs both of which are regularly embedded in actual trading.

(b) In this context, we set up the concept of a transactionally efficacious market where arbitrage takes place not only when prices converge to boundary values as in the standard efficiency model, but within a “trap set” whose structure duly was appraised

(c) The Transactionally Efficient Market arises from a transactionally efficacious market with rational expectations. This seems a natural way of extending the standard efficient market model when transaction costs, mediators and microstructure enter the picture.

(d) Arbitrage gaps arising from current prices, fundamental values and transaction costs, lead to a vectorial function in charge of coping whenever arbitrage be performed against the “trap set”.

(e) A meaningful outgrowth regarding the dynamic arbitrage gap could be stated as follows:

- Although price trajectories patterned with market microstructure and transaction costs make rather uncommon the sort of point-wise convergence towards a specific walrasian value as claimed by the standard model, the "trap set" behaves as an attractor of price trajectories whenever the market is transactionally efficacious.

- But it is within the trap set, which precludes chances to attain supernormal profits most of the time, that an intense dynamics evolves. The transition from price trajectories out of the "trap set" into this region is functionally managed by the vectorial arbitrage gap function.

- The complex environment this model conveys allows for nonlinear dynamics, with chaotic patterns of behavior eventually, either for price trajectories or arbitrage gaps for bid-ask prices, inclusive of transaction costs and microstructure.

(f) Modelling the dynamic arbitrage gap as a deterministic system seems fairly realistic so as to understand the financial assets price mechanism in a deeper way. Besides, the quasi-stochastic behavior in prices that comes out of chaotic trajectories, should make the model suitable for simulation trials, and this is a promising direction for further applied research.

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